Asymptotics of sharp constants of Markov-Bernstein inequalities in integral norm with classical weights

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Overview

1. Markov-Bernstein inequalities
2. Markov-Bernstein inequality in inner product spaces.
3. Hermite weight
4. Laguerre weight
5. Jacobi weight

Markov-Bernstein inequalities.

A.A. Markov *On one question of D.I.Mendeleev*, Izvestiya Peterburg Akademii Nauk, 62 (1889), pp. 1–24, (in Russian).
For any polynomial Q, $\deg Q \leq n$ one has

$$||Q'||_{C[-1,1]} \leq n^2 ||Q||_{C[-1,1]}$$

where

$$||Q||_{C[-1,1]} = \max_{-1 \leq x \leq 1} |Q(x)|$$

The constant n^2 is sharp (Chebychev polynomials).

Markov-Bernstein inequalities.

S.N. Bernshtein *On the best approximation of the continuous functions by means of polynomials with fixed degree*, Soobsheniya Kharkovskogo Matem. Obshestva (1912), (in Russian).
For any polynomial Q, $\deg Q \leq n$ one has

$$||Q'||_{C(\Delta)} \leq n||Q||_{C(\Delta)}$$

where

$$||Q||_{C(\Delta)} = \max_{|z|\leq 1} |Q(z)|$$

The constant n is sharp $(Q(z) = z^n)$.

Markov-Bernstein inequalities. General setting.

Let $\mathcal{P}_n$ be the set of polynomials of degree at most n, X be a metric space and for any n, $\mathcal{P}_n \subset X$. For a given n find the sharp constant in inequality

$$||Q'||_X \leq M_n ||Q||_X, \quad \deg Q \leq n$$

Many generalizations and applications of Markov-Bernstein inequalities are known.
G.V. Milovanović, D.S. Mitrinović, Th.M. Rassias, *Topics in polynomials: extremal problems, inequalities, zeros*, World Scientific, Singapore 1994.

Recent applications: Markov-Bernstein inequalities are primary tools to prove approximate degree lower bounds on Boolean functions.
M. Bun J.Thaler *Dual Lower Bounds for Approximate Degree and Markov-Bernstein Inequalities*, arXiv:1302.6191v3, 22 Mar 2014

Markov-Bernstein inequality in inner product spaces.

Suppose X is an inner product functional space, with inner product (f,g). Then one can construct the sequence of orthonormal polynomials π_n such that $(\pi_k, \pi_l) = \delta_{k,l}, \quad k, l = 0, 1, 2, \ldots$. In this case one has for any $Q \in \mathcal{P}_n$

$$\begin{array}{ll} Q = u_0\pi_0 + u_1\pi_1 + \ldots + u_n\pi_n, & ||Q||^2 = |u_0|^2 + |u_1|^2 + \ldots + |u_n|^2 \\ Q' = v_0\pi_0 + v_1\pi_1 + \ldots + v_{n-1}\pi_{n-1}, & ||Q'||^2 = |v_0|^2 + |v_1|^2 + \ldots + |v_{n-1}|^2 \end{array}$$

The sharp constant in Markov-Bernstein inequality is

$$M_n^2 = \sup_{\deg Q \le n} \frac{||Q'||^2}{||Q||^2}$$

It is sufficient to consider the subspace with $u_0 = 0$ $(|u_0|^2 + |u_1|^2 + \ldots + |u_n|^2 \ge |u_1|^2 + \ldots + |u_n|^2)$. In this case the linear transformation $(u_1, u_2, \ldots, u_n) \to (v_0, v_1, \ldots, v_{n-1})$ is bijective on R^n.

Markov-Bernstein inequality in inner product spaces.

Denote by A the matrix of transformation $v = Au$, $u = (u_1, u_2, \ldots, u_n)$, $v = (v_0, v_1, \ldots, v_{n-1})$. Then one has

$$M_n = \sup_{u \neq 0} \frac{||Au||}{||u||} = ||A|| = \sqrt{\lambda_{\max}(AA^T)}$$

The matrix A (matrix of differential operator in the basis of orthonormal polynomials) is crucial in the study of the sharp constants in Markov-Bernstein inequality.

Remark: In some cases matrix $B = A^{-1}$ is more appropriate to use. One has

$$M_n = \frac{1}{\sqrt{\lambda_{\min}(B^T B)}}$$

Markov-Bernstein inequality in inner product spaces. Summary

ONP basis π_n in X : $(\pi_k, \pi_l) = \delta_{k,l}, \quad k, l = 0, 1, 2, \ldots$.

$\forall Q \in \mathcal{P}_n$ take

$$Q = \sum_{k=0}^{n-1} u_k \pi_{k+1}, \qquad Q' = \sum_{k=0}^{n-1} y_k \pi_k$$

Define A : $y = Au$, $u = (u_1, \ldots, u_n)$, $y = (y_0, \ldots, y_{n-1})$ or $B = A^{-1}$.
Then

$$M_n = \sup_{u \neq 0} \frac{||Au||}{||u||} = ||A|| = \sqrt{\lambda_{\max}(AA^T)} = \frac{1}{\sqrt{\lambda_{\min}(B^T B)}}$$

Hermite weight

Inner product

$$(f,g) = \int_{-\infty}^{+\infty} f(x)g(x)e^{-x^2}\,dx$$

Polynomials π_k are orthonormal Hermite polynomials. The following relations are known

$$\pi_k' = \sqrt{2k}\;\pi_{k-1}, \quad k = 1, 2, 3, \dots$$

In this case matrix A is diagonal

$$A = \operatorname{diag}(\sqrt{2}, \sqrt{4}, \dots, \sqrt{2n})$$

Therefore the sharp constant in Markov-Bernstein inequality for Hermite weight is $M_n = \sqrt{2n}$ (E.Schmidt, 1944)

Laguerre weight $e^{-x}dx$

Polynomials π_k are orthonormal Laguerre polynomials. We have

$$B^T B = \begin{bmatrix} 1 & 1 & \cdots & 0 & 0 \\ 1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & \cdots & 1 & 2 \end{bmatrix}$$

Characteristic polynomials is perturbed (co-recursive) Chebyshev poly.

$$\Delta_n = (\lambda - 2)\Delta_{n-1} - \Delta_{n-2}, \;\; \Delta_1 = \lambda - 1, \Delta_2 = \lambda^2 - 3\lambda + 1$$

For the eigenvalues of $B^T B$, one can obtain

$$\lambda_j(B^T B) = 4 \sin^2 \frac{(2j-1)\pi}{4n+2}, \;\; j = 1, 2, \dots, n$$

Therefore

$$M_n = \frac{1}{2 \sin \frac{\pi}{4n+2}} = \frac{2n}{\pi} [1 + o(1)]$$

P.Turan [1960]

Generalized Laguerre weight $x^\alpha e^{-x} dx$

$$B^T B = \begin{bmatrix} \alpha_0 & \beta_1 & 0 & \cdots & 0 & 0 \\ \beta_1 & \alpha_1 & \beta_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-2} & \beta_{n-1} \\ 0 & 0 & 0 & \cdots & \beta_{n-1} & \alpha_{n-1} \end{bmatrix}$$

where $\alpha_0 = (1+\alpha)$, $\alpha_k = (2 + \frac{\alpha}{k+1})$, $\beta_k^2 = 1 + \frac{\alpha}{k}$, $k = 1, 2, \ldots, n-1$.
Characteristic polynomials $\Delta_k(\lambda)$ satisfy the recurrence equation

$$\Delta_k = (\lambda - 2 - \frac{\alpha}{k})\Delta_{k-1} - (1 + \frac{\alpha}{k-1})\Delta_{k-2}, \quad k \geq 2$$

with initial conditions

$$\Delta_1 = \lambda - \alpha - 1, \quad \Delta_2 = \lambda^2 - (3 + \frac{3}{2}\alpha)\lambda + (1+\alpha)(1 + \frac{\alpha}{2}).$$

Polynomials $\Delta_k(\lambda)$ are orthogonal with respect to some measure with the support on $[0, 4]$. Therefore one needs asymptotics of this sequence of polynomials in the neighborhood of the point 0.

Generalized Laguerre weight - Asymptotics

$$\Delta_k = (\lambda - 2 - \frac{\alpha}{k})\Delta_{k-1} - (1 + \frac{\alpha}{k-1})\Delta_{k-2}, \qquad k \geq 2$$

Let $q_n(\lambda) = \Delta_n(\lambda)/\Delta_n(0)$, $\lambda = h^2$, $nh = z$, $q_n^h = q_n(h^2)$.

We check $\Delta_n(0)/\Delta_{n+1}(0) = 1 + o(\frac{1}{n})$ then $n \to \infty$, $z \in K \Subset \mathbb{C}$

$$\frac{q_{n+1}^h - 2q_n^h + q_{n-1}^h}{h^2} + \frac{2\alpha}{z}\frac{q_{n+1}^h - q_{n-1}^h}{2h} + q_n^h = o(\frac{1}{n})$$

It can be proved [Apt., Sb. Math. 1993 76 35–50], $\quad h \to 0$

$$q_n(\frac{z^2}{n^2}) = j_\nu(z)\left[1 + o(\frac{1}{n})\right], \qquad j_\nu(z) = 2^\nu \Gamma(\nu+1) J_\nu(z),$$

where $J_\nu(z)$ is the Bessel function, $\nu = \frac{\alpha-1}{2}$.

Generalized Laguerre weight

Theorem: let $\nu = \frac{\alpha-1}{2}$. Then one has for the Markov-Bernstein best constant for generalized Laguerre weight

$$M_n = \frac{n}{z_1}\left[1 + o(1)\right]$$

where z_1 is the zero of the Bessel function $J_\nu(z)$, nearest to the origin. In particular, if $\alpha = 0$ then $\nu = -\frac{1}{2}$ and z_1 is the zero of the Bessel function

$$J_{-1/2}(z) = \sqrt{\frac{2\pi}{z}}\ \cos(z)$$

nearest to the origin. One has $z_1 = \pi/2$

A.I. Aptekarev, A. Draux and V.A. Kaliaguine , *On asymptotics of the exact constants in the Markov-Bernshtein inequalities with classical weighted integral metrics*, Uspekhi. Mat. Nauk, 55, (2000) 173–174;

Jacobi weight

Inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)(1-x)^{\alpha}(1+x)^{\beta}dx$$

First known case. Legendre weight $\alpha = \beta = 0$

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx$$

E.Schmidt [1944].

$$M_n = \frac{(2n+3)^2}{4\pi}[1+o(1)] = \frac{n^2}{\pi}[1+o(1)]$$

Jacobi weight

This case requires more deep results on local asymptotics for the polynomial solutions of recurrence relations.
First step. Generalized eigenvalue problem

$$\mathbf{A}y = \lambda \mathbf{D}y,$$

where the matrix $\mathbf{D}$ is diagonal and matrix $\mathbf{A}$ is five diagonal. To find an eigenvector y one has to solve a five terms finite difference equation (FD)

$$\lambda f_k y_k = a_k y_{k+2} + b_k y_{k+1} + c_k y_k + d_k y_{k-1} + e_k y_{k-2}, k \geq 0$$

with initial and boundary conditions

$$y_{-1} = y_{-2} = 0, \qquad y_n = y_{n+1} = 0. \quad !$$

Here all coefficients $a_k, b_k, c_k, d_k, e_k, f_k$ of finite difference equation are rational functions on k.

$$M_n^2 = \lambda_{\min}^{-1}(\mathbf{A}, \mathbf{D}) \; ,$$

where $\lambda_{\min}(\mathbf{A}, \mathbf{D})$ is a root (with the minimal modulus) of the equation

Step 1 tools

$$\frac{d}{dx} P_k^{(\alpha,\beta)}(x) = k P_{k-1}^{(\alpha+1,\beta+1)}(x) \,,$$

$$P_n^{(\alpha,\beta)} = P_n^{(\alpha+1,\beta)} - \frac{2n(n+\beta) P_{n-1}^{(\alpha+1,\beta)}}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} \,,$$

$$\|P_n^{(\alpha,\beta)}\|^2 = (P_n^{(\alpha,\beta)}, P_n^{(\alpha,\beta)}) = 2^{2n+\alpha+\beta-1} \frac{n!(n+\alpha)!(n+\beta)!(n+\alpha+\beta)!}{(2n+\alpha+\beta)!(2n+\alpha+\beta+1)!} .$$

Step 1 summary

Thus we have (**A** - 5 diagonals, **D** - 1 diagonal)

$$M_n^2 = \lambda_{\min}^{-1}(\mathbf{A}, \mathbf{D}) \; ,$$

where $\lambda_{\min}(\mathbf{A}, \mathbf{D})$ is a root (with the minimal modulus) of the equation

$$\det(\mathbf{A} - \lambda \mathbf{D}) = 0 \; ,$$

and correspondingly the eigenvector $\vec{y}$

$$(\mathbf{A} - \lambda_{\min} \mathbf{D})\vec{y} = 0.$$

It gives a 5-term recurrence relation

$$[\,(\mathbf{A} - \lambda \mathbf{D})\; \vec{y}\,]_k = 0 \; , \qquad k = 0, \ldots, n-1 \, ,$$

with boundary condition

$$y_{-1} = y_{-2} = 0, \qquad y_n = y_{n+1} = 0. \quad !$$

Step 1 the result

$$y_k =: x_k \frac{(2k+\alpha+\beta+1)!}{2^k (k+\alpha)!(k+\beta)!}\,, \qquad k = 0, \dots, n-1\,.$$

$$x_{k+2} \tfrac{(k+2)!}{(k-2)!} \tfrac{(2k+\alpha+\beta)!}{(2k+\alpha+\beta-3)!} \tfrac{(k+\alpha+\beta+1)!}{(k+\alpha+\beta-1)!} \;=\; x_{k+1} \tfrac{(k+1)!}{(k-2)!} \tfrac{(2k+\alpha+\beta-1)!}{(2k+\alpha+\beta-3)!}\,\Xi_1 +$$

$$x_k \left(\Xi_2 - \lambda\, \tfrac{(k+1)!}{(k-2)!} \tfrac{(2k+\alpha+\beta+4)!}{(2k+\alpha+\beta-3)!} \tfrac{(k+\alpha+\beta)}{4}\right) + x_{k-2} \tfrac{(k+1)!}{(k-1)!} \tfrac{(2k+\alpha+\beta+4)!}{(2k+\alpha+\beta+1)!} \tfrac{(k+\alpha)!}{(k+\alpha-2)!} \tfrac{(k+\beta)!}{(k+\beta-2)!}$$

$$+ \quad x_{k-1} \tfrac{(2k+\alpha+\beta+4)!}{(2k+\alpha+\beta+2)!} (k^2-1)(k+\alpha)(k+\beta)(2k+\alpha+\beta-1)(\alpha+\beta-2)(\alpha-\beta)\,,$$

where

$$\Xi_1 = (k+\alpha+\beta)(2k+\alpha+\beta+3)(\alpha+\beta-2)(\alpha-\beta),\ \Xi_2 = \frac{k^4}{2} + k^3(1+\alpha+\beta)$$

$$+\ k^2\ \tfrac{2\alpha+2\beta+2\alpha^2+3\alpha\beta+2\beta^2+1}{2}\ +\ k\ \tfrac{(1+\alpha+\beta)(\alpha^2+\alpha\beta+\beta^2)}{2}\ +\ O_{\alpha,\,\beta}(1).$$

Jacobi weight Matching a DE

It is more convenient to work with first order difference equation for a vectors $Y_k = (y_{k-2}, y_{k-1}, y_k, y_{k+1})^T$ which can be written in the form

$$\frac{Y_{k+2} - Y_k}{2/n} = \frac{n}{k} M(k,\lambda) Y_k, \;\; k \geq 0, \qquad Y_0 = (0,0,C_1,C_2)^{Tr}$$

Second step. We choose an appropriate scaling and go to the limit in (FD) to obtain a system of differential equations (DE).
Scaling: $\lambda = \frac{z}{n^4}$, $\frac{k}{n} \to t$, $n \to \infty$. System of DE has a general solution

$$y(t,z) = (y_1(t,z), y_2(t,z), 2ty_1'(t,z), 2ty_2'(t,z))^T$$

where $y_j(t,z)$ are a linear combinations of Bessel functions $J_\nu(\sqrt{z}\frac{t^2}{2})$ and $Y_\nu(\sqrt{z}\frac{t^2}{2})$

Jacobi weight Step 2 more details

$$\vec{y} = (y_1, y_2, y_3, y_4)^{Tr} \quad \vec{y}(t,z) := \lim_{\substack{n\to\infty \\ \frac{k}{n}\to t}} Y_k(\lambda)\Big|_{\lambda = z/n^4}$$

$$\frac{d}{dt}\vec{y}(t,z) = \frac{1}{t}\widetilde{\mathbf{M}}(t,z)\,\vec{y}(t,z), \quad \widetilde{\mathbf{M}}(t,z) = \lim_{\frac{k}{n}\to t} \mathbf{M}(k, \frac{z}{n^4})\ ,$$

where

$$\widetilde{\mathbf{M}}(t,z) = \begin{pmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \\ -2zt^4 + 2\alpha(\alpha-2) & 0 & 2 & 0 \\ 0 & -2zt^4 + 2\beta(\beta-2) & 0 & 2 \end{pmatrix}.$$

$$\frac{d^2}{dt^2} y_j(t,l) = \frac{1}{t}\frac{d}{dt} y_j(t,l) - \left(t^2 l - \frac{b_j(b_j-2)}{t^2}\right) y_j\ , \quad b_j := \begin{cases} \alpha, & j = 1 \\ \beta, & j = 2 \end{cases}.$$

Jacobi weight Step 2 the results

The General Solution (GS) $\quad y_j(t,z),\ j=1,2$ is

$$\widetilde{C}_{1,j}\, t\, J_{\nu(b_j)}\left(\sqrt{z}\,\frac{t^2}{2}\right)+\widetilde{C}_{2,j}\, t\, Y_{\nu(b_j)}\left(\sqrt{z}\,\frac{t^2}{2}\right), \quad \nu(b) := \frac{b-1}{2}.$$

GS of the DE problem

$$\vec{y}(t,z)=\big(\ y_1(t,z),\, y_2(t,z),\, 2ty_1'(t,z),\, 2ty_2'(t,z)\ \big)^{Tr}.$$

Approximate GS of the FD problem $C_1\overrightarrow{Y}_k^{(1)}+C_2\overrightarrow{Y}_k^{(2)}$, for $k\in\mathbb{Z}$:

$$\overrightarrow{Y}_k^{(1)}(\lambda)\approx\begin{pmatrix} y_1\left(\frac{k}{n},\lambda n^4\right)\\ 0\\ \frac{2k}{n}y_1'\left(\frac{k}{n},\lambda n^4\right)\\ 0\end{pmatrix}, \quad \overrightarrow{Y}_k^{(2)}(\lambda)\approx\begin{pmatrix} 0\\ y_2\left(\frac{k}{n},\lambda n^4\right)\\ 0\\ \frac{2k}{n}y_2'\left(\frac{k}{n},\lambda n^4\right)\end{pmatrix},$$

here ($'$) denotes the derivative with respect to the first variable.

$$\begin{cases} n\to\infty\\ \dfrac{k}{n}\to t\in K\Subset(0,1]\end{cases},$$

Jacobi weight Matching BC

Step 3. Now we need to match the boundary conditions of (DE) with left boundary conditions of (FD). This is the rather sophisticated part of work.

1) We construct two Particular Solutions (PS) of the FD problem for $\lambda = 0$ which fit BC at the left end;

2) We construct two PS of the DE problem matching when $t \to 0$ PS of the FD problem, it gives for $\lambda = z\, n^{-4}, \; t = k/n, \; n \to \infty$

$$\overrightarrow{Y}_k^{(j)}(z\, n^{-4}) = n^{b_j} \vec{y}^{(j)}(\frac{k}{n}, z) + o\left(k^{b_j}\right) \; .$$

Finally the right end BC (determinant)

$$J_{\nu(\alpha)}\left(\frac{\sqrt{z}}{2}\right) \; J_{\nu(\beta)}\left(\frac{\sqrt{z}}{2}\right) + o(1) = 0$$

give approximate values of eigenvalues $\lambda = \frac{z}{n^4}$. Here $\nu(\alpha) = (\alpha - 1)/2$.

Jacobi weight Asymptotics Result

Theorem

Let $\{\overrightarrow{Y}_k^{(j)}(\lambda)\}$ *be a PS of the FD problem:* $\overrightarrow{Y}_0^{(j)}(\lambda) = \overrightarrow{Y}_0^{(j)}(0)$, $j = 1, 2$.
Let the Jacobi weight parameters (α, β) *satisfy the condition:*

$$|\alpha - \beta| < 4 , \qquad \alpha, \beta > -1 .$$

Then for $\lambda = \frac{z}{n^4}$, $\frac{k}{n} \to t$, $n \to \infty$, *uniformly for* $z \in \widetilde{K} \Subset \mathbb{C}$, $t \in K \Subset (0, 1]$

$$\overrightarrow{Y}_k^{(j)}(\lambda) = n^{b_j} \left(\vec{y}^{(j)}(t, z) + o(t^{b_j}) \right) , \quad b_j = \begin{cases} \alpha, & j = 1 \\ \beta, & j = 2 \end{cases} .$$

Here $\vec{y}^{(j)}$ *are PS of the DE problem satisfying the matching condition*

$$\vec{y}^{(j)}(t, z) = t^{b_j} \left(\overrightarrow{C}_0^{(j)} + o\,(1) \right), \quad \overrightarrow{C}_0^{(1)} := \begin{pmatrix} 2 \\ 0 \\ 4\alpha \\ 0 \end{pmatrix}, \overrightarrow{C}_0^{(2)} := \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4\beta \end{pmatrix} .$$

Jacobi weight the result

Theorem

Let $\nu = \min\{\frac{\alpha-1}{2}, \frac{\beta-1}{2}\}$ *and* $|\alpha - \beta| < 4$*. Then one has for the Markov-Bernstein best constant for Jacobi weight*

$$M_n = \frac{n^2}{2z_1}\,[1 + o(1)]$$

where z_1 *is the zero of the Bessel function* $J_\nu(z)$*, nearest to the origin.*

If $\alpha = \beta = 0$ then $\nu = -\frac{1}{2}$ and z_1 is the zero of the Bessel function

$$J_{-1/2}(z) = \sqrt{\frac{2\pi}{z}}\ \cos(z).$$

One has $z_1 = \pi/2$ and $M_n = \frac{n^2}{\pi}\,[1 + o(1)]$ which is E.Schmidt's result.

A.Aptekarev A.Draux V.Kalyagin D.Tulyakov *Asymptotics of sharp constants of Markov-Bernstein inequalities in integral norm with Jacobi weight*, accepted for Proceedings of AMS, 2013

Remark on the technical condition

$$|\alpha - \beta| < 4 \qquad ?$$

THANK YOU FOR YOUR ATTENTION!