Asymptotics of weighted Bergman polynomials

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$$\int_\Omega P_n \overline{P}_k w \, dm = \delta_{n,k},$$

where the leading coefficient is normalized to be positive:

$$P_n(z) = \kappa_n z^n + a_{n-1}^{(n)} z^{n-1} + \ldots + a_0^{(n)}, \qquad \kappa_n > 0.$$

Here, dm stands for Lebesgue measure.

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Here, dm stands for Lebesgue measure. Note we only consider absolutely continuous measures of orthogonality.

Background

When $w \equiv 1$, orthonormal polynomials were studied on Jordan domains by Bochner, Carleman [1922], Bergman [1950], Fuks [1951], Rosenbloom& Warschawski [1955], Smirnov& Lebedev [1964].

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Closely connected to these works is the issue of the density of polynomials in the holomorphic Bergman space that was investigated by Keldys [1939], Markusevic& Farell [1942], Dzrbasjan [1948], Mergelyan [1962], Saginjaw.

Background cont'd

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- Saff, Stahl, Stylianopoulos and Totik [2014] deal with multiply connected analytic domains (archipelogoos with lakes).

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- Other incentives come from Heele-Shaw flows, particle systems, ...

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Korovkin [1947] obtained exterior and interior asymptotics for the case of a simply connected analytic domain Ω when the weight is of the form $|\Phi' g|^2$ in a neighborhood of $\partial\Omega$, where $\Phi : \overline{\mathbb{C}} \setminus \overline{\Omega} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is the conformal map with $\Phi'(\infty) > 0$ and g is holomorphic nonvanishing in a neighborhood of $\overline{\mathbb{C}} \setminus \Omega$.
The result reads

$$P_n(z) = \left(\frac{n+1}{\pi}\right)^{1/2} \frac{\Phi^n(z)}{g(z)}(1 + O(\lambda^n)), \qquad 0 \leq \lambda < 1,$$

for z in a neighborhood of $\overline{\mathbb{C}} \setminus \Omega$.

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$$P_n(z) = \left(\frac{n+1}{\pi}\right)^{1/2} \Phi^n(z)\Phi'(z)S^-(z)\left(1 + O\left((\log n/n)^\alpha\right)\right)$$

where α is the Hölder exponent of w and

$$S^-(z) = \exp\left\{\frac{1}{4\pi}\int_{\mathbb{T}} \frac{e^{i\theta}+\Phi(z)}{e^{i\theta}-\Phi(z)} \log w(\Phi^{-1}(e^{i\theta}))\, d\theta\right\}$$

is the exterior Szegő function of $w_{|\partial\Omega}$.

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- In fact $S_{w_1}^-$ is the largest (in modulus) nonvanishing analytic function in $\overline{\mathbb{C}} \setminus \overline{\Omega}$ whose nontangential maximal function lies in $L^2(\partial\Omega)$ and whose nontangential limit on $\partial\Omega$ has squared modulus $1/w_1$ a.e..

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- The interior Szegő function $S_{w_1}^+(z)$ is defined similarly for $z \in \Omega$ using the interior conformal map Φ_1, and this time $|S_{w_1}^+|^2 = w_1$ on $\partial\Omega$.
- $S_{w_1}^{\pm}$ solve a "Riemann-Hilbert problem":

$$S_{w_1}^-(\xi) = \left(\overline{S_{w_1}^+(\Phi_1^{-1}\circ\Phi(\xi))}\right)^{-1}, \qquad \xi \in \partial\Omega.$$

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- Simanek [2012] obtained ratio asymptotics for large $|z|$ and analytic simply connected Ω, for weights which are conformal images of certain product measures on the unit disk $\mathbb{D}$:

$$w = \Big(\nu(\theta) \times \tau(\rho)\Big) \circ \varphi, \qquad \varphi : \Omega \to \mathbb{D}.$$

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- Mina-Diaz and Simanek [2013] gave necessary conditions on w for exterior asymptotics to hold.

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- Saff and Simon speculated that ratio asymptotics exists for $|z|$ large, as soon as w does not vanish too much in a neighborhood of $\partial\Omega$, at least for reasonably smooth Ω (generalization of a theorem by Rakhmanov on the circle).
- Defining what "does not vanish too much" means is part of the question.

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- In fact all these results can be thought of as perturbations of the 1-D case, where the influence of the "germ" of the weight close to the boundary asymptotically dominates all other phenomena.
- It is to ensure this dominancy that nonzeroing assumptions on w to the boundary $\partial\Omega$ are made.

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Assumptions

- Ω is an analytic Jordan domain. In particular, $\Psi := \Phi^{-1}$ extends conformally into a map from $\{|z| > 1 - \varepsilon\}$ onto $\overline{\mathbb{C}} \setminus \overline{\Omega}_1$, where $\overline{\Omega}_1 \subset \Omega$.

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- Putting $\Psi_r(e^{i\theta}) := \Psi(re^{i\theta})$, we assume that $w \circ \Psi_r$ converges in $L^p(\mathbb{T})$ as $r \to 1$, for some $p > 1$. If F is the limit, we put $w_1 := F \circ \Phi$.

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- Putting $\Gamma_\eta := \Psi(\{|z| = \eta\})$ for $1 - \varepsilon < \eta < 1$, we assume that

$$\sup_{1-\varepsilon<\eta<1} \int_{\Gamma_\eta} \log^- w \, \log^+(\log^- w) \, d\sigma < +\infty.$$

This last condition expresses that the weight does not vanish too much in the vicinity of $\partial\Omega$.

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Theorem

Under the previous assumptions it holds that

$$P_n(z) = \left(\frac{n+1}{\pi}\right)^{1/2} \Phi^n(z)\Phi'(z)S_{w_1}^-(z)(1+o(1))$$

locally uniformly outside the convex hull of Ω, with $S_{w_1}^-$ the exterior Szegő function of w_1.

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- When $\{z_k\}$ is dense in Ω, then w vanishes in the neighborhood of every point.

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- At this point, we will know that

  $$\liminf_{n\to+\infty} \frac{\kappa_n}{\sqrt{n+1}} = \left(\pi \mathcal{G}_{w_1}\right)^{-1/2},$$

  where $\mathcal{G}_{w_1} = \exp\{\int_{\mathbb{T}} \log(w_1 \circ \Psi)\}$ is the geometric mean.

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- Having at our disposal a sequence of polynomials Q_n with dominant coefficient $\alpha_n \sim \kappa_n$ whose $L^2(w)$ norm is asymptotically 1, we use a technique of Widom:

$$\|P_n - Q_n\|^2_{L^2(w)} = \|P_n\|^2_{L^2(w)} + \|Q_n\|^2_{L^2(w)} - 2\Re\langle P_n, Q_n\rangle_w$$

$$= 1 + \|Q_n\|^2_{L^2(w)} - 2\frac{\alpha_n}{\kappa_n} \to 0.$$

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$$|P_n/Q_n - 1| \leq \|P_n - Q_n\|_{L^2(w)} \mathrm{d}(z, \mathrm{Conv}\Omega) + \mathrm{diam}\Omega)^2/\mathrm{d}^2(z, \mathrm{Conv}\Omega),$$

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- Finally one checks by inspection that

$$Q_n(z) = \left(\frac{n+1}{\pi}\right)^{1/2} z^n S^-_{w_1}(z)\{1 + o(1)\}, \qquad z \notin \overline{\Omega}.$$

A closer look at the upper bound

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Theorem

For Ω an analytic Jordan domain and $w \geq 0$ a weight function in $L^1(\Omega)$, it holds that

$$\limsup_{n\to\infty} \kappa_n \frac{(\operatorname{cap}\Omega)^{n+1}}{\sqrt{n+1}} \leq \frac{1}{\sqrt{\pi}\left(\operatorname{ess\,sup}_{r\to 1^-}\ G_{w\circ\Psi_r}^{1/2}\right)}$$

where cap indicates the logarithmic capacity.

A closer look at the upper bound cont'd

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Proof:

Let $A_{1,R}$ to be the annular region between Γ_1 and Γ_R, $R < 1$, and consider the integral:

$$J_n := \int_R^1 rdr \int_0^{2\pi} e^{-2ni\theta} \left(P_n \circ \Psi_r(e^{i\theta}) \Psi'(re^{i\theta}) / S^-_{w\circ\Psi_r}(e^{i\theta}) \right)^2 d\theta.$$

On the one hand, it holds that

$$\begin{aligned} |J_n| &\le \int_R^1 rdr \int_0^{2\pi} |P_n(\Psi(re^{i\theta})|^2 w(\Psi(re^{i\theta}))|\Psi'(re^{i\theta})|^2 \, d\theta \\ &= \int_{A_{1,R}} |P(\xi)|^2 w(\xi) \, dm(\xi) \le 1. \end{aligned}$$

A closer look at the upper bound cont'd

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Proof cont'd:
On the other hand, using the residue formula at infinity for Hardy functions of class $H^1(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$, we get

$$J_n = 2\pi \int_R^1 r^{2n+1}\, dr \frac{1}{2i\pi} \int_{\mathbb{T}_r} \left(\frac{P_n(\Psi(\xi))}{\xi^n S^-_{w\circ\Psi_r}(\xi)} \right)^2 \frac{d\xi}{\xi}.$$

$$= 2\pi \kappa_n^2 (\operatorname{cap}\Omega)^{2n+2} \int_R^1 r^{2n+1} G_{w\circ\Psi_r}\, dr.$$

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Finally, it is elementary that

$$\limsup_{n\to\infty} (2n+2)^{-1} \int_R^1 r^{2n+1} G_{w\circ\Psi_r}\, dr \leq \operatorname{ess} \sup_{r\to 1^-} G^{1/2}_{w\circ\Psi_r}.$$

A closer look at the lower bound

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Theorem

Let $w \in L^1(\mathbb{D})$ and assume that

$$w_1 := \lim_{r \to 1^-} w_r \;\; \text{exists} \;\; \text{in} \;\; L^p(\mathbb{T}), \quad p > 1.$$

Then

$$\liminf_{n \to +\infty} \frac{\kappa_n}{\sqrt{n+1}} \geq (\pi \mathcal{G}_{w_1})^{-1/2},$$

where the right-hand side may be finite or infinite depending whether $\int_{\mathbb{T}} \log w_1 > -\infty$ or $\int_{\mathbb{T}} \log w_1 = -\infty$.

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$$\kappa_n = \sup\{\kappa;\ \ \exists\ P(z) = \kappa z^n + a_{n-1} z^{n-1} + \cdots + a_0,\ \ \|P\|_{L^2(w)} \leq 1\}.$$

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The proof rests on the construction of a sequence of auxiliary polynomial whose leading coefficient matches the lower bound and whose norm in $L^2(w)$ is asymptotically 1. Such a sequence is given by

$$Q_n(e^{i\theta}) := \left(\frac{n+1}{\pi}\right)^{1/2} e^{(n-k_n)i\theta}\mathbf{P}_+\left(e^{ik_n\theta} S_{w_1,+}^{-1}(e^{-i\theta})\right).$$

Here $\mathbf{P}_+$ indicates analytic projection that selects Fourier coefficients of non-negative index, and $k_n \to \infty$ but $k_n/n \to 0$.

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- To remove the assumption that $w \geq \delta > 0$, we apply the preceding case to $w^{\{m\}} := w + \delta_m$ where $\delta_m \in (0,1) \to 0$ and we use that κ_n increases when the measure decreases.

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- To remove the assumption that $w \geq \delta > 0$, we apply the preceding case to $w^{\{m\}} := w + \delta_m$ where $\delta_m \in (0,1) \to 0$ and we use that κ_n increases when the measure decreases.
- Besides, the needed convergence

$$\lim_{m\to\infty} \mathcal{G}_{w_1^{\{m\}}} = \mathcal{G}_{w_1}$$

follows easily from dominated and monotone convergence applied to the positive and negative parts of the functions.

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we get by Cauchy's theorem:

$$F_n(z) = \Phi^n(z)\Phi'(z) + \frac{1}{2i\pi}\int_{\Gamma_R}\frac{\Phi^n(\xi)\Phi'(\xi)}{\xi - z}\,d\xi, \qquad z \in V_R.$$

Then, a straightforward majorization gives us

$$\left|F_n(z) - \Phi^n(z)\Phi'(z)\right| \leq CR^n, \qquad z \in \ V_R. \quad R > R_0,$$

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- Previous estimates on F_n and our choice of k_n make $\mathfrak{Q}_n \to 0$ locally uniformly in Ω.
- Moreover, change of variable shows that

$$\limsup_{n\to\infty} \|\mathfrak{Q}_n\|_{L^2(\Omega\cap V_{R_1}, w)} \leq \limsup_{n\to\infty} \|Q_n\|_{L^2(\mathcal{A}_R, w\circ\psi)} \leq 1.$$

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  Lemma. Let h_k be a bounded sequence in $\mathfrak{h}^1$ that converges pointwise a.e. to h on $\mathbb{T}$. Then $h \in \mathfrak{h}^1$ and $h_k d\theta$ converges weak-* to $h d\theta$ in $\mathcal{M}$.

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Lemma. Let h_k be a bounded sequence in $\mathfrak{h}^1$ that converges pointwise a.e. to h on $\mathbb{T}$. Then $h \in \mathfrak{h}^1$ and $h_k d\theta$ converges weak-* to $hd\theta$ in $\mathcal{M}$.
Here $\mathcal{M}$.is the space of complex measure and $\mathfrak{h}^1$ is the real Hardy space. For positive functions, $h \in \mathfrak{h}^1$ is equivalent to $h \log^+ h \in L^1(\mathbb{T})$ y a theorem of Riesz and Zygmund.

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(asymptotic conformality on $\partial\Omega$). This is enough to control the surface integral contribution due to $\bar{\partial}\Psi$ when deforming integration from $\partial\Omega$ to Γ_R.

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- Can one obtain rates?
- The techniques can also be used to give examples where κ_n has no limit, hence there are are strong asymptotics. Can one produce examples where there are no ratio asymptotics?

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Thank You !!