Asymptotics of Jacobi-Type Orthogonal Polynomials

The Case of the Cross.

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Abstract

We investigate asymptotic behavior of polynomials $Q_n(z)$ satisfying non-Hermitian orthogonality relations

$$\int_\Delta s^k Q_n(s)\rho(s)\mathrm{d}s = 0, \quad k \in \{0,\dots,n-1\}, \tag{1}$$

where $\Delta := [-a,a] \cup [-\mathrm{i}b, \mathrm{i}b]$, $a, b > 0$, and $\rho(s)$ is a Jacobi-type weight. The primary motivation for this work is study of the convergence properties of the Padé approximants to functions of the form

$$f(z) = (z-a)^{\alpha_1}(z-\mathrm{i}b)^{\alpha_2}(z+a)^{\alpha_3}(z+\mathrm{i}b)^{\alpha_4},$$

where the exponents $\alpha_i \notin \mathbb{Z}$ add up to an integer.

Class $\mathcal{W}_\ell$

Let ℓ be a positive integer or infinity. We shall say that a function $\rho(s)$ on Δ belongs to the class $\mathcal{W}_\ell$ if

(i) $\rho_i(s) := \rho_{|\Delta_i}(s)$ factors as a product $\rho_i(s) = \rho_i^*(s)(s-a_i)^{\alpha_i}$, where the function $\rho_i^*(z)$ is non-vanishing and holomorphic in some neighborhood of Δ_i, $\alpha_i > -1$, and $(z-a_i)^{\alpha_i}$ is a branch holomorphic across $\Delta \setminus \{a_i\}$, $i \in \{1,2,3,4\}$;

(ii) in some neighborhood of the origin it holds that $(\rho_1\rho_3)(z) = c(\rho_2\rho_4)(z)$ for some constant c;

(iii) it holds that $\rho_1(0) + \rho_2(0) + \rho_3(0) + \rho_4(0) = 0$;

(iv) the quantities $\rho_i^{(l)}(0)/\rho_i(0)$, $0 \le l < \ell$, do not depend on $i \in \{1,2,3,4\}$.

Padé Approximation

For an integrable weight $\rho(s)$ on Δ define

$$\widehat{\rho}(z) := \frac{1}{2\pi\mathrm{i}}\int_\Delta \frac{\rho(s)\mathrm{d}s}{s-z}, \quad z \in \overline{\mathbb{C}} \setminus \Delta. \tag{2}$$

In particular, it can be readily verified that the functions

$$\sum_{i=1}^4 C_i \log(z-a_i) \quad \text{and} \quad \prod_{i=1}^4 (z-a_i)^{\alpha_i},$$

where the constants C_i add up to zero and the exponents $-1 < \alpha_i \notin \mathbb{Z}$ add up to an integer, possess branches holomorphic off Δ that can be represented by (2) for certain weight functions in $\mathcal{W}_\infty$ (the second function can represented by (2) up to an addition of a polynomial).

Given $\widehat{\rho}(z)$ as in (2), it follows from the orthogonality relations (1) that there exists a polynomial $P_n(z)$ of degree at most $n-1$ such that

$$R_n(z) := (Q_n\widehat{\rho})(z) - P_n(z) = \mathcal{O}(z^{-n-1}) \quad \text{as} \quad z \to \infty. \tag{3}$$

The rational function $[n/n]_{\widehat{\rho}}(z) := P_n(z)/Q_n(z)$ is called the n-th diagonal Padé approximant to $\widehat{\rho}(z)$.

Baker-Akhiezer Functions

To state the main results requires the introduction of functions $\Psi_n^{(j)}(z)$, known as *Baker - Akhiezer* functions. Their main properties are:

- $\Psi_n^{(j)}$ are sectionally meromorphic on the Riemann surface $\mathfrak{R}$ of $w^2 = (z^2-a^2)(z^2+b^2)$.
- $\Psi_n^{(0)}$ is geometrically large on closed subsets of the complement of Δ.
- $\Psi_n^{(0)}$ has the divisor $(n-1)\infty^{(1)} - \mathcal{D}_n - n\infty^{(0)}$, where $\mathcal{D}_n$ is a positive divisor of degree 1.
- In this setting, $\mathcal{D}_n$ depends only on the parity of n. It will be important that $\mathcal{D}_n \neq \infty^{(0)}$
- This can be guaranteed along one of three subsequences (depending only on ρ): $\mathbb{N}_\rho = \mathbb{N}, 2\mathbb{N}, \mathbb{N}\setminus 2\mathbb{N}$.

Main Result

Choose branches of the logarithm such that

$$\nu := \frac{1}{2\pi\mathrm{i}}\sum_{i=1}^4 (-1)^i \log\rho_i(0) \quad \text{satisfies } \mathrm{Re}(\nu) \in \left(-\frac{1}{2}, \frac{1}{2}\right]. \tag{4}$$

Let $\rho(s) \in \mathcal{W}_\ell$, where ℓ is a positive integer or infinity, be such that $\mathrm{Re}(\nu) \in (-1/2, 1/2)$. Then it holds for all $n \in \mathbb{N}_\rho$ large enough that

$$Q_n(z) = \gamma_n\big(1 + \upsilon_{n1}(z)\big)\Psi_n\big(z^{(0)}\big) + \gamma_n\upsilon_{n2}(z)\Psi_{n-1}\big(z^{(0)}\big) \tag{5}$$

locally uniformly in $\overline{\mathbb{C}} \setminus \Delta$, where γ_n is given explicitly in terms of theta functions, $\upsilon_{ni}(\infty) = 0$, and

$$\upsilon_{ni}(z) = \mathcal{O}\big(n^{-d_{\nu,\ell}}\big), \quad d_{\nu,\ell} := \left(\frac{1}{2} - |\mathrm{Re}(\nu)|\right)\frac{\ell - 2|\mathrm{Re}(\nu)|}{\ell + 1 - 2|\mathrm{Re}(\nu)|},$$

locally uniformly in $\overline{\mathbb{C}} \setminus \{0\}$ (uniformly in $\overline{\mathbb{C}}$ when $\ell = \infty$). In particular, polynomials $Q_n(z)$ have degree n for all $n \in \mathbb{N}_\rho$ large enough.

Riemann-Hilbert Problem

Just as was first done by Fokas, Its, and Kitaev [2, 3], we connect the orthogonal polynomials $Q_n(z)$ to a 2×2 matrix Riemann-Hilbert problem. To this end, suppose that the index n is such that

$$\deg Q_n = n \quad \text{and} \quad R_{n-1}(z) \sim z^{-n} \quad \text{as} \quad z \to \infty, \tag{6}$$

where $R_n(z)$ is given by (3). Furthermore, let

$$\boldsymbol{Y}(z) := \begin{pmatrix} Q_n(z) & R_n(z) \\ k_{n-1}Q_{n-1}(z) & k_{n-1}R_{n-1}(z) \end{pmatrix}, \tag{7}$$

where k_{n-1} is a constant such that $k_{n-1}R_{n-1}(z) = z^{-n}(1 + o(1))$ near infinity. Then $\boldsymbol{Y}(z)$ solves the following Riemann-Hilbert problem (RHP-$\boldsymbol{Y}$):

(a) $\boldsymbol{Y}(z)$ is analytic in $\mathbb{C} \setminus \Delta$ and $\lim_{z\to\infty} \boldsymbol{Y}(z)z^{-n\sigma_3} = \boldsymbol{I}$.

(b) $\boldsymbol{Y}(z)$ has continuous traces on Δ° that satisfy

$$\boldsymbol{Y}_+(s) = \boldsymbol{Y}_-(s)\begin{pmatrix} 1 & \rho(s) \\ 0 & 1 \end{pmatrix}, \quad s \in \Delta^\circ.$$

(c) $\boldsymbol{Y}(z)$ is bounded around the origin. Near a_i's the first column is bounded, while the second column is bounded for $\alpha_i > 0$, possesses a logarithmic singularity for $\alpha_i = 0$, and a power singularity (i.e. $\sim |z-a_i|^{\alpha_i}$) for $\alpha_i > 0$.

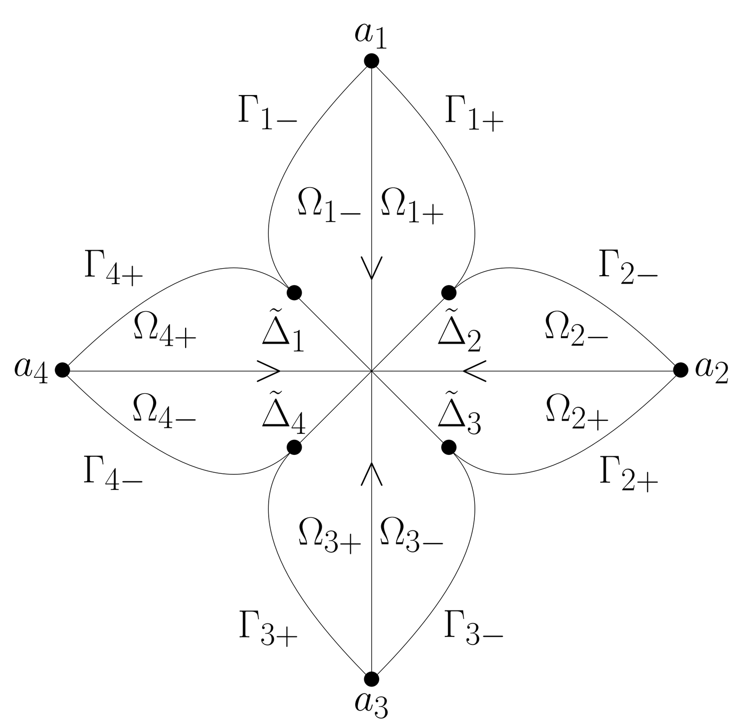

Figure 1: The opening of the "lenses."

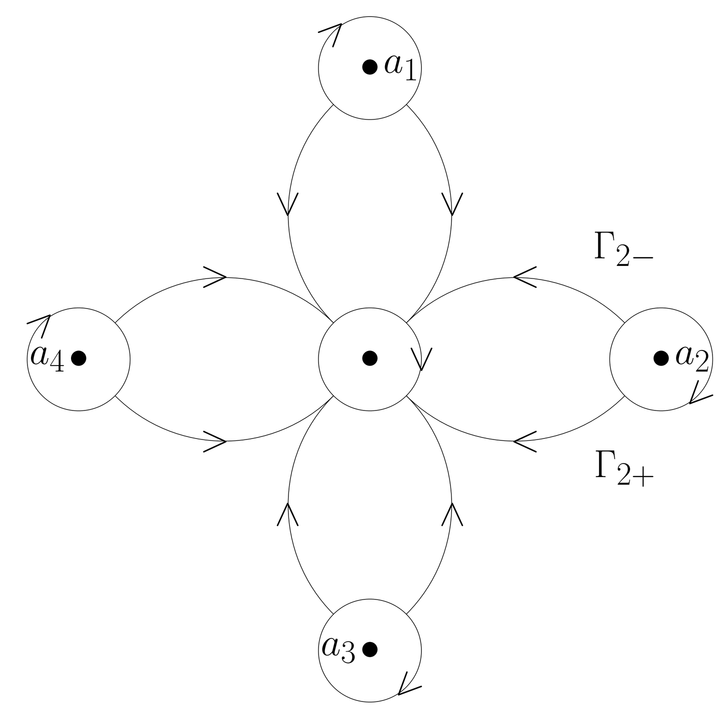

Figure 2: Contour for final Riemann-Hilbert problem.

Model Problem

A key step in the analysis is to identify a matrix function, denoted $\boldsymbol{\Psi}_{s_1,s_2}(\zeta)$, that satisfies the following jump conditions:

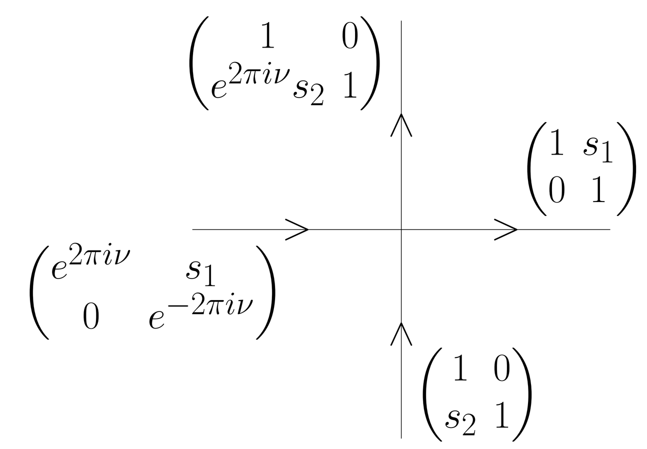

Figure 3: The jumps of $\boldsymbol{\Psi}_{s_1,s_2}(\zeta)$ over the corresponding contours.

Here, $s_1, s_2 \in \mathbb{C}$ are independent parameters related to $\nu \in \mathbb{C}$, $\mathrm{Re}(\nu) \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ by

$$e^{-2\pi i\nu} := 1 - s_1 s_2 \tag{8}$$

(we slightly abuse the notation here as the parameter ν has already been fixed in (4); however, we shall use the construction below with parameters s_1, s_2 such that (8) holds with ν from (4)). Define constants b, d by

$$b := -s_1\frac{\Gamma(1+\nu)}{\sqrt{2\pi}} \quad \text{and} \quad d := -s_2 e^{\nu\pi i}\frac{\Gamma(1-\nu)}{\sqrt{2\pi}},$$

where $\Gamma(z)$ is the standard Gamma function. Denote by $D_\mu(\zeta)$ the parabolic cylinder function in Whittaker's notations. Then, the matrix function $\boldsymbol{\Psi}_{s_1,s_2}(\zeta)$ is given by

$$\begin{pmatrix} D_\nu(2\zeta) & bD_{-\nu-1}(-2i\zeta) \\ dD_{\nu-1}(2\zeta) & D_{-\nu}(-2i\zeta) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{-\pi i\nu/2} \end{pmatrix}, \qquad \arg(\zeta) \in \left(0, \tfrac{\pi}{2}\right),$$

$$\begin{pmatrix} D_\nu(-2\zeta) & bD_{-\nu-1}(-2i\zeta) \\ -dD_{\nu-1}(-2\zeta) & D_{-\nu}(-2i\zeta) \end{pmatrix}\begin{pmatrix} e^{\pi i\nu} & 0 \\ 0 & e^{-\pi i\nu/2} \end{pmatrix}, \quad \arg(\zeta) \in \left(\tfrac{\pi}{2}, \pi\right),$$

$$\begin{pmatrix} D_\nu(-2\zeta) & -bD_{-\nu-1}(2i\zeta) \\ -dD_{\nu-1}(-2\zeta) & D_{-\nu}(2i\zeta) \end{pmatrix}\begin{pmatrix} e^{-\pi i\nu} & 0 \\ 0 & e^{\pi i\nu/2} \end{pmatrix}, \quad \arg(\zeta) \in \left(-\tfrac{\pi}{2}, -\pi\right),$$

$$\begin{pmatrix} D_\nu(2\zeta) & -bD_{-\nu-1}(2i\zeta) \\ dD_{\nu-1}(2\zeta) & D_{-\nu}(2i\zeta) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i\nu/2} \end{pmatrix}, \qquad \arg(\zeta) \in \left(0, -\tfrac{\pi}{2}\right).$$

Case $\mathrm{Re}(\nu) = \frac{1}{2}$

In this case, (5) holds for $n \in \mathbb{N}_\rho^*$ large enough, where $\mathbb{N}_\rho^*$ is a further restricted (necessarily so!) to indices that satisfy the condition (all functions are explicitly known):

$$\mathbb{N}_\rho^*(\varepsilon) := \left\{ n \in \mathbb{N}_\rho : \left| \left(\frac{ab}{2n}\right)^{\mathrm{Im}(\nu)} - (-1)^{\iota_{j(n)}} S^2(0)\Phi\big(\boldsymbol{z}_{j(n)}\big)\Phi^{2(n-1)}\big(0^{(1)}\big) \right| > \varepsilon \right\}.$$

In this case, (5) holds with

$$\upsilon_{ni}(z) = \frac{L_{ni}}{z} + \mathcal{O}\left(n^{-d_{\nu,\ell}}\right),$$

where L_{ni} are bounded and the error estimate holds locally uniformly in $\overline{\mathbb{C}} \setminus \{0\}$ (uniformly in $\overline{\mathbb{C}}$ when $\ell = \infty$).

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