Universality for zeros of random polynomials

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Kac polynomials

A Kac polynomial on the complex plane is of the form

$$f_N(z) = \sum_{j=0}^{N} a_j z^j$$

We assume that a_j's are real or complex identically distributed independent i.i.d. random variables and let $\mathbf{P}$ denote their distribution law. Identifying

$$Poly_N \to \mathbb{C}^{N+1}$$

$$f_N \to (a_j)_{j=0}^{N}$$

we obtain the probability space $(Poly_N, Prob_N)$ where $Prob_N$ is the $(N+1)$-fold product probability measure induced from $\mathbf{P}$.

Then we from the product probability space $\prod_{N=1}^{\infty}(Poly_N, Prob_N)$ whose elements are sequences of random polynomials.

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Empirical measure of zeros

Writing

$$f_N(z) = \sum_{j=0}^{N} a_j z^j = a_N \prod_{j=1}^{N} (z - \zeta_j)$$

where ζ_j's are the roots of f_N. We may define a random variable

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$$f_N \to [\mathcal{Z}_{f_N}] := \sum_{j=1}^{N} \delta_{\zeta_j}.$$

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and we define the expected zero measure

$$\langle \mathbb{E}[\mathcal{Z}_{f_N}], \varphi \rangle := \int_{Poly_N} \sum_{j=1}^{N} \varphi(\zeta_j) dProb_N$$

for continuous function $\varphi \in \mathcal{C}_c(\mathbb{C})$.

Kac Ensemble: Gaussian Case

Theorem (Kac-Hammersley-Shepp-Vanderbei)

Assume that a_j are i.i.d. complex (or real) valued Gaussian random variables of mean zero and variance one. Then

- $\frac{1}{N}\mathbb{E}[\mathcal{Z}_{f_N}] \to \frac{1}{2\pi}d\theta$ *weakly as* $N \to \infty$.
- *Almost surely* $\frac{1}{N}\mathcal{Z}_{f_N} \to \frac{1}{2\pi}d\theta$ *weakly as* $N \to \infty$.

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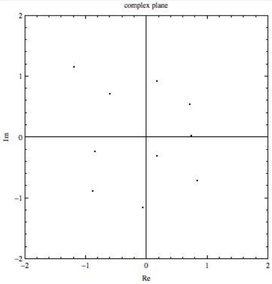

f_{10}.

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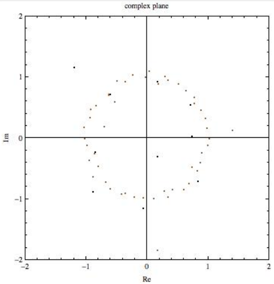

f_{10}, f_{50}

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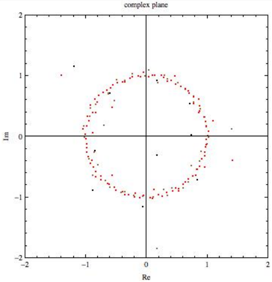

f_{10}, f_{50}, f_{100}

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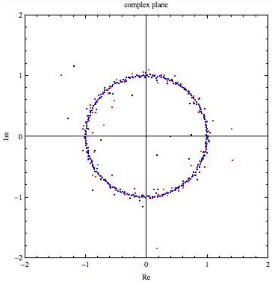

$f_{10}, f_{50}, f_{100}, f_{250}$

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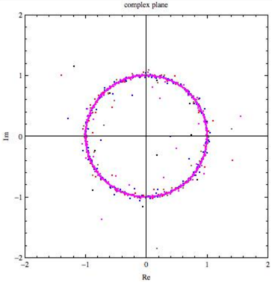

$f_{10}, f_{50}, f_{100}, f_{250}, f_{500}$

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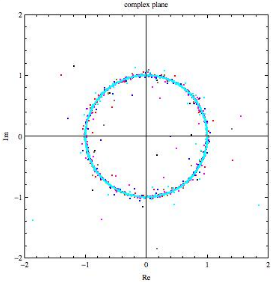

$f_{10}, f_{50}, f_{100}, f_{250}, f_{500}, f_{1000}$

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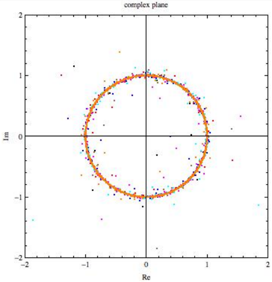

$f_{10}, f_{50}, f_{100}, f_{250}, f_{500}, f_{1000}, f_{2000}$

Why?

Monomials z^j for an ONB for $Poly_N$ relative to

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(z)\overline{g(z)} d\theta$$

and Bergman kernel

$$K_N(z, w) = \sum_{j=0}^{N} z^j \overline{w}^j$$

is reproducing kernel of point evaluation at z that is

$$f(z) = \int_{S^1} f(w) K_N(z, w) \frac{d\theta}{2\pi}$$

and $K(z, z) = \frac{1-|z|^{2n+2}}{1-|z|^2}$. Moreover, $K_N(e^{i\theta}, e^{i\theta}) = N + 1$.

Proof of Gaussian case

This implies that

$$\frac{1}{2N}\log K_N(z,z) \to \log^+|z| := \max(\log|z|, 0)$$

locally uniformly on $\mathbb{C}$. Therefore $\Delta(\frac{1}{2N}\log K_N(z,z)) \to \frac{d\theta}{2\pi}$

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Now,

$$\frac{1}{N}\log|f_N(z)| = \frac{1}{N}\log|\langle a^N, u^N(z)\rangle| + \frac{1}{2N}\log K_N(z,z)$$

where

$$\langle a^N, u^N(z)\rangle = \sum_j a_j \frac{z^j}{\sqrt{K_N(z,z)}}$$

and $u^N(z) = (\frac{1}{\sqrt{K_N(z,z)}}, \frac{z}{\sqrt{K_N(z,z)}}, \dots, \frac{z^N}{\sqrt{K_N(z,z)}}) \in \mathbb{C}^{N+1}$ is a unit vector.

Proof of Gaussian case

For a test function $\varphi \in C_c(\mathbb{C})$ by using $\Delta(\frac{1}{N}\log|f_N|) = \mathcal{Z}_{f_N}$

$$\begin{aligned}
\langle \frac{1}{N}\mathbb{E}[\mathcal{Z}_{f_N}], \varphi\rangle &= \int_{\mathbb{C}^{N+1}} \langle \Delta(\frac{1}{2N}\log K_N(z,z)), \varphi\rangle dProb_N \\
&+ \int_{\mathbb{C}^{N+1}} \langle \Delta(\frac{1}{N}\log|\langle a^N, u^N(z)\rangle|), \varphi\rangle dProb_N \\
&= \langle \Delta(\frac{1}{2N}\log K_N(z,z)), \varphi\rangle \\
&+ \int_{\mathbb{C}^{N+1}} \langle \frac{1}{N}\log|\langle a^N, u^N(z)\rangle|, \Delta\varphi\rangle dProb_N \\
&= \langle \Delta(\frac{1}{2N}\log K_N(z,z)), \varphi\rangle \\
&+ \int_{\mathbb{C}} \Delta\varphi \left(\frac{1}{N}\int_{\mathbb{C}^{N+1}} \log|\langle a^N, u^N(z)\rangle| dProb_N\right)
\end{aligned}$$

Proof of Gaussian case

Since $u^N(z)$ is a unit vector by unitary invariance of Gaussian we obtain

$$\begin{aligned}\langle \frac{1}{N}\mathbb{E}[\mathcal{Z}_{f_N}], \varphi\rangle &= \langle \Delta(\frac{1}{2N}\log K_N(z,z)), \varphi\rangle \\ &+ \frac{1}{N}\int_{\mathbb{C}} \Delta\varphi(\frac{1}{\pi}\int_{\mathbb{C}} \log|a_0|e^{-|a_0|^2} d\lambda(a_0))\end{aligned}$$

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Hence the result follows.

Universality for Kac Ensemble

Theorem (Ibragimov and Zaporozhets 13')

If the coefficients a_j are non-degenerate i.i.d. random variables then $\mathbb{E}[\log^+ |a_j|] < \infty$ is necessary and sufficient for $\frac{1}{N}\mathcal{Z}_{f_N} \xrightarrow{w} \frac{1}{2\pi}d\theta$.

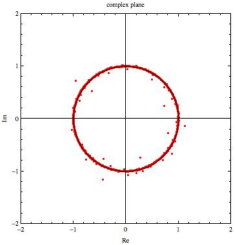

Figure: Standard Gaussian

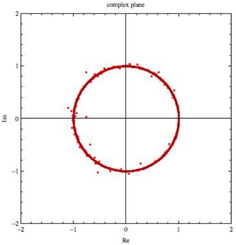

Figure: Uniform Distribution

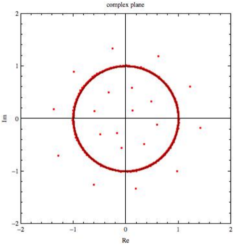

Figure: Cauchy Distribution

$f_N(z) = \sum_{j=0}^{N} a_j z^j$ is a random polynomial of degree 1000.

Complex Geometry

Let X be a projective manifold of dimension m and $L \to X$ be a holomorphic line bundle. We say that L is positive if L admits a smooth Hermitian metric h whose curvature form ω_h is a Kähler form. We denote the induced volume form $dV_h = \frac{1}{m!}\omega_h^m$. Denote by $H^0(X, L)$ the vector space of global holomorphic sections. We define a L^2-norm on $H^0(X, L)$ by

$$\|s\|_h^2 = \int_X |s|_h^2 dV_h$$

We consider tensor powers $L^{\otimes N}$ endowed with the metric $h_N := h^{\otimes N}$.

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We consider tensor powers $L^{\otimes N}$ endowed with the metric $h_N := h^{\otimes N}$. For a fixed orthonormal basis $\{S_j^{(N)}\}$ of $H^0(X, L^{\otimes N})$ the N^{th} Bergman kernel

$$K_N(x, y) = \sum_j S_j^{(N)}(x) \otimes \overline{S_j^{(N)}(y)}$$

is the integral kernel of the projection $\mathcal{C}^\infty(X, L^{\otimes n}) \to H^0(X, L^{\otimes N})$.

Motivation

- Value Distribution Theory
- PDEs
- Algebraic Geometry

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Model Example

Example

Let $X = \mathbb{P}^m$ be complex projective space and $L = \mathcal{O}(1)$ hyperplane bundle. Then $H^0(\mathbb{P}^m, \mathcal{O}(N))$ can be identified with homogenous polynomials in $m+1$ variables of degree N. Letting $h = h_{FS}$ Fubini-Study metric, the sections

$$S_J = \left[\frac{(N+m)!}{m! j_0! \dots j_m!}\right]^{\frac{1}{2}} z^J, \quad J = (j_0, \dots, j_m), |J| = N$$

form ONB for $H^0(\mathbb{P}^m, \mathcal{O}(N))$

SU(m+1) Polynomials are defined by

$$f_N(z_0, \dots, z_m) = \sum_{|J|=N} \frac{a_J}{\sqrt{j_0! \dots j_m!}} z^J$$

where a_J are iid standard complex Gaussian.

Random Holomorphic Sections

A random holomorphic section $s_n \in H^0(X, L^{\otimes N})$ is of the form

$$s_n = \sum_j a_j S_j^{(n)}$$

where the coefficients a_j are iid copies of a non-degenerate real or complex random variable ζ. Then we endow $H^0(X, L^{\otimes N})$ with a $d_N := \dim(H^0(X, L^{\otimes N}))$ fold product probability measure $Prob_N$.

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Problem

Are zeros of random holomorphic sections uniformly distributed relative to a deterministic measure? If such a measure exists, is it independent of the choice of the law of random coefficients?

Zeros of Sections

Denote by $Z_{s_N^1,\dots,s_N^k} := \{z \in X : s_N^1(z) = \cdots = s_N^k(z) = 0\}$.

Theorem (Bertini)

For generic sections $s_N^1,\dots,s_N^k$ the zero sets $Z_{s_N^j}$ are smooth and intersect transversally. In particular, simultaneous zero set $Z_{s_N^1,\dots,s_N^k}$ is a complex submanifold of codimension k.

We denote by $\mathcal{Z}_{s_N^1,\dots,s_N^k}$ the current of integration along the variety $Z_{s_N^1,\dots,s_N^k}$ Note that

$$\langle \mathcal{Z}_{s_N^1,\dots,s_N^k}, \omega_h^{m-k} \rangle = n^k c_1(L)^m$$

where $c_1(L)^m := \int_X \omega_h^m$. In particular,

$$\frac{1}{n^k}\mathcal{Z}_{s_N^1,\dots,s_N^k} \text{ is cohomologous to } \omega_h^k.$$

Gaussian Case

Theorem (Shiffman-Zelditch 99)

Assume a_j are iid complex Gaussian with mean zero variance one. Then for each $1 \leq k \leq m$

$$\mathbb{E}[\frac{1}{N^k}\mathcal{Z}_{s_N^1,\dots,s_N^k}] \to \omega_h^k$$

Moreover, almost surely,

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Proof.

(1) $\mathbb{E}[\mathcal{Z}_{s_N}] = w_N$ where $w_N := \Phi_N^* w_{FS}$ and $\Phi_N : X \to \mathbb{P}^{d_N}$ is the Kodaira map. Thus first part follows from Bergman kernel asymptotics of Tian-Catlin-Zelditch which implies $\omega_N \to \omega_h$ as $N \to \infty$.

Proof of Shiffman-Zelditch Theorem

Proof.

(2) For almost everywhere convergence we need a variance estimate: for fixed test form φ

$$Var[|\langle \frac{1}{N}\mathcal{Z}_{s_N}, \varphi\rangle] := \mathbb{E}[|\langle \frac{1}{N}\mathcal{Z}_{s_N}, \varphi\rangle - \langle \omega_N, \varphi\rangle|^2] = O(N^{-2})$$

Letting $Y_N(s_N) := (\langle \frac{1}{N}\mathcal{Z}_{s_N}, \varphi\rangle - \langle \omega_N, \varphi\rangle)^2$ and observing

$$\int_{H^0(X,L^{\otimes N})} \sum_N Y_N(s_N) dProb_N = \sum_N \int_{H^0(X,L^{\otimes N})} Y_N(s_N) dProb_N$$
$$< \infty$$

implies the assertion. □

Distribution of Zeros

A weighted compact set (K, ϕ) is a pair of a compact set $K \subset X$ and a weight ϕ of a continuos Hermitian metric $e^{-2\phi}$ on L. Then we define equilibrium weight

$$V_{K,\phi} := \sup\{\psi \text{ psh weight on } L : \psi \leq \phi \text{ on } K\}$$

We say that K is regular if $V_{K,\phi}$ is continuous.

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Theorem (Guedj-Zeriahi, Berman-Boucksom)

If K is non-pluripolar compact set then $V^*_{K,\phi}$ is a psh weight on L. Its curvature current $T_{K,\phi}$ is a positive closed current on X representing the first Chern class $c_1(L) \in H^{1,1}(X, \mathbb{R})$.

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Example

If $K = X$ and $\phi = 0$ is the flat metric then $T_{K,\phi} = \omega_h$

Distribution of Zeros

For a measure ν supported on K we denote

$$\|s_N\|_2^2 = \int_K |s|_\phi^2 d\nu$$

Definition (BM measure)

A triple (K, ϕ, ν) satisfies Bernstein-Markov (BM) inequality if there exists $M_N > 0$ such that

$$\sup_{x \in K} |s_N(x)|_{\phi_N} \leq M_N \|s_N\|_2 \text{ for every } s_N \in H^0(X, L^{\otimes N})$$

and $\limsup_{N\to\infty} M_N^{\frac{1}{N}} = 1$.

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and $\limsup_{N\to\infty} M_N^{\frac{1}{N}} = 1$.

Theorem (Bloom-Shiffman, B13)

If K is regular and (K, ϕ, ν) satisfies (BM) inequality then $\frac{1}{2N} \log K_N(x, x)$ converges locally uniformly to $V_{K,\phi}$

Non-Gaussian Ensembles

- More generally, Dinh & Sibony 06' studied equidistribution problem endowing $SH^0(X, L^{\otimes N})$ with moderate measures (which are locally Monge-Ampère measure of a Hölder continuous psh function). They used a new method based on formalism of meromorphic transforms (still uses Bergman kernel asymptotics).

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- Bloom & Levenberg 13' proved convergence of expected zero currents $N^{-k}\mathbb{E}[\mathcal{Z}_{s_N^1,\dots,s_N^k}]$ for polynomially decaying distributions i.e.

$$\mathbf{P}\{z \in \mathbb{C} : |z| > R\} = O(R^{-2})$$

and posed almost sure convergence of $N^{-k}\mathcal{Z}_{s_N^1,\dots,s_N^k}$ as an open problem.

Universality of Zeros

Theorem (B13')

Assume that a_j are iid real or complex random variables whose distribution law $\mathbf{P}$ *has a bounded density and*

$$\mathbf{P}\{z \in \mathbb{C} : \log|z| > R\} = O(R^{-\rho}) \text{ as } R \to \infty$$

for some $\rho > m+1$. Then for each $1 \leq k \leq m$ the expected current of zeros

$$N^{-k}\mathbb{E}[Z_{s_N^1,\dots,s_N^k}] \to T^k_{K,\phi}$$

in the sense of currents as $N \to \infty$.

Universality of Zeros

Theorem (B13')

Assume that a_j are iid real or complex random variables whose distribution law **P** *has a bounded density and*

$$\mathbf{P}\{z \in \mathbb{C} : \log|z| > R\} = O(R^{-\rho}) \text{ as } R \to \infty$$

for some $\rho > m+1$. Then for each $1 \leq k \leq m$ the expected current of zeros

$$N^{-k}\mathbb{E}[Z_{s_N^1,\dots,s_N^k}] \to T^k_{K,\phi}$$

in the sense of currents as $N \to \infty$.Moreover, if the ambient space X is complex homogeneous then almost surely

$$N^{-k}[Z_{s_N^1,\dots,s_N^k}] \to T^k_{K,\phi}$$

in the sense of currents as $N \to \infty$.

Sketch of Proof

Proof is based on induction on k. For $k=1$ and fixed smooth form φ we write

$$\mathbb{E}[\frac{1}{N}\langle \mathcal{Z}_{s_N},\varphi\rangle] = I_N^1(z) + I_N^2(z)$$

where $\langle I_N^1(z),\varphi\rangle = \langle (dd^c(\frac{1}{2N}\log K_N(z,z)),\varphi\rangle \to \langle T_{K,\phi},\varphi\rangle$ as $N\to\infty$

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$$\langle I_N^2(z), \varphi\rangle = \int_{H^0(X,L^{\otimes N})} \langle u_N(z), dd^c\varphi\rangle dProb_N$$

Lemma

$$|I_N^2(z)| = O(N^{m+1-\rho})$$

This proves the first part.

Sketch of Proof

To prove almost sure convergence, we need a variance estimate

Lemma

Assuem that X is complex homogenous.

$$Var[\frac{1}{N}\langle \mathcal{Z}_{s_N}, \varphi \rangle] = O(N^{m+1-\rho})$$

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Lemma

Assuem that X is complex homogenous.

$$Var[\frac{1}{N}\langle \mathcal{Z}_{s_N}, \varphi\rangle] = O(N^{m+1-\rho})$$

Then by Kolmogorov's law of large numbers with probability one

$$\frac{1}{N}\sum_{j=1}^{N}\langle \frac{1}{j}\mathcal{Z}_{s_j}, \varphi\rangle \to \langle T_{K,\phi}, \varphi\rangle$$

Then using a classical lemma from dynamics $\exists$ a subsequence N_j of density one such that $\langle \frac{1}{N_j}\mathcal{Z}_{s_{N_j}}, \varphi\rangle \to \langle T_{K,\phi}, \varphi\rangle$. Finally, by continuity of potentials we conclude that the whole sequence converge.

SU(2) Polynomials

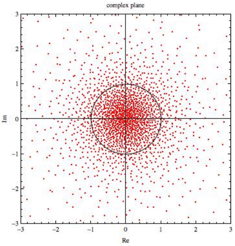

Figure: Standard Gaussian

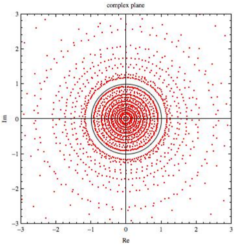

Figure: Cauchy Distribution

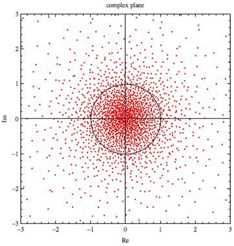

Figure: Bernoulli Distribution

The figures illustrate zero distribution of a random $SU(2)$ polynomial $f_N(z) = \sum_{j=0}^{n} a_j \sqrt{\binom{n}{j}} z^j$ of degree 2000.

Universality in codimension one

Denote by volume of hypersurface Z_{s_n} in an open set $U \subset X$ by $Vol_{2m-2}^{\omega_h}(Z_{s_n} \cap U)$ and

$$\mathcal{V}_U := \frac{1}{(m-1)!} \int_U T_{K,\phi} \wedge (\omega_h)^{m-1}.$$

Theorem (B15')

For every open set $U \subset X$ such that ∂U has zero volume

$$Prob\left\{ s_N : \lim_{n\to\infty} \frac{1}{N} Vol_{2m-2}^{\omega_h}(Z_{s_N} \cap U) = \mathcal{V}_U \right\} = 1$$

if and only if

$$\mathbb{E}[\log^+ |a_j|] < \infty.$$

From holomorphic sections to orthogonal polynomials

Dehomogenizing:

If $X = \mathbb{P}^m$ and $L = \mathcal{O}(1)$ then the open set on $\mathbb{C}^m \subset \mathbb{P}^m$ every $s_N \in H^0(\mathbb{P}^m, \mathcal{O}(N))$ can be written as

$$s_N = f_N \sigma^{\otimes N}$$

where σ is a holomorphic section whose zero set $Z_\sigma = \mathbb{P}^m - \mathbb{C}^m$ and f_N is a polynomial of total degree at most N.

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where σ is a holomorphic section whose zero set $Z_\sigma = \mathbb{P}^m - \mathbb{C}^m$ and f_N is a polynomial of total degree at most N. In particular, the point-wise norm of s_N on $\mathbb{C}^m$ relative to a continuous metric $e^{-2\phi}$ on $\mathcal{O}(1)$ becomes

$$|s_N(z)|^2_\phi = |f_N(z)|^2 e^{-2NQ(z)}$$

where Q is a continuous function defined by

$$Q(z) := \phi(z) - h_{FS} := \phi(z) - \frac{1}{2}\ln(1 + |z|^2).$$

Here, h_{FS} denotes the weight function of Fubini-Study metric which is characterized (up to a constant) by its invariance under $SU(m)$.

Orthogonal polynomials

Assuming that ν is BM measure supported on a regular compact set $K \subset \mathbb{C}^m$ the current geometric setting reduces to random orthogonal polynomials

$$f_N(z) = \sum_{j=1}^{d_N} a_j F_j^{(N)}(z)$$

where $\{F_j^{(n)}\}$ form an ONB relative to

$$\langle f, g \rangle := \int_K f(z)\overline{g(z)} e^{-NQ(z)} d\nu$$

Examples:

- $K = S^1$ and $Q \equiv 0$ & $\nu = \frac{1}{2\pi} d\theta$ gives Kac polynomials.
- $K = \mathbb{P}^1$ and $Q(z) = \frac{1}{2}\log(1 + |z|^2)$ & $\nu = \frac{dz}{\pi(1+|z|^2)^2}$ for $z \in \mathbb{C}$ gives Elliptic polynomials.

Super logarithmic growth

Let $K \subset \mathbb{C}^m$ be a non-pluripolar (possibly unbounded) closed set and $Q : K \to \mathbb{R}$ be a continuous functoin satisfying

$$Q(z) \geq (1+\epsilon)\ln|z| \text{ for } z \gg 1$$

for some $\epsilon > 0$. This implies that $\|f_N\|_2 < \infty$ for every polynomial f_N.

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Example (Model case)

$K = \mathbb{C}$ and $Q(z) = |z|^2$ then it is well-known that $T_{K,Q} = 1_{\mathbb{D}} d\lambda(z)$

Theorem

The Monge-Ampère measure

$$\mu_{K,Q} := T_{K,Q} \wedge \cdots \wedge T_{K,Q}$$

has compact support and $supp(\mu_{K,Q}) \subset \{z \in K : Q(z) = V^*_{K,Q}(z)\}$

Weyl Polynomials, Circular law

If $K = \mathbb{C}$, $Q(z) = \frac{|z|^2}{2}$ and $\nu = d\lambda$ Lebesgue measure on $\mathbb{C}$. A random Weyl polynomial is of the from

$$W_N(z) = \sum_{j=0}^{N} a_j \sqrt{\frac{N^j}{j!}} z^j.$$

Theorem (Kabluchko-Zaporozhets 12', B15')

Assume that a_j are i.i.d. non-degenerate real or complex valued random variables. The logarithmic moment

$$\mathbb{E}[\log(1 + |a_j|)] < \infty.$$

if and only if

$$\mathbb{P}\Big\{ W_N : \frac{1}{N}\mathcal{Z}_N(U, W_n) \xrightarrow[N\to\infty]{} \frac{1}{\pi}\lambda(U \cap \mathbb{D}) \Big\} = 1$$

for every open set $U \Subset \mathbb{C}$ such that ∂U has zero Lebesgue measure.

Asymptotic Normality of Smooth Statistics

Theorem (Sodin-Tsirelson 04')

Let $f_N(z) = \sum_{j=0}^{N} a_j z^j$ with a_j are i.i.d. Gaussian random variables and ψ be function of class $\mathcal{C}^3$. Then the random variables

$$\mathcal{X}_N(\psi) := \frac{\langle Z_{f_N}, \psi\rangle - \mathbb{E}(\langle Z_{f_N}, \psi\rangle)}{\sqrt{Var\langle Z_{f_N}, \psi\rangle}}$$

converge in distribution to $\mathcal{N}(0,1)$ as $N \to \infty$.

Asymptotic normality of zeros were also obtained by Maslova 74' for random polynomials with real i.i.d. coefficients a_j such that $\mathbb{E}a_j = 0$ and $\mathbb{E}|a_j|^{2+\epsilon} < \infty$.

Problem

Do linear statistics $\mathcal{X}_N(\psi)$ of multivariable random polynomials or random holomorphic sections enjoy asymptotic normality?

Asymptotic Normality of Smooth Statistics

- Sodin and Nazarov revisited Gaussian analytic functions and improved Sodin-Tsirelson
- Shiffman-Zelditch prove asymptotic normality for zeros of Gaussian random holomorphic sections in codimension one.
- B15' (in preparation) Gaussian random holomorphic sections for $\mathcal{C}^2$-metrics and smoothly bounded domains

Problem

What about higher codimensions?

Thank you!