Hyperuniformity in the compact setting: measuring the fine structure of a sequence of point sets on the sphere

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6. Oct 2018

Midwestern Workshop on Asymptotic Analysis 2018

Indiana University, Bloomington, USA

Graz University of Technology

FWF Der Wissenschaftsfonds.

UNIFORMITY ON THE SPHERE

(X_N) is **a.u.d. on** $\mathbb{S}^d$ if

$$\lim_{N\to\infty} \frac{\#\{k : \mathbf{x}_{k,N} \in B\}}{N} = \sigma_d(B)$$

for every Riemann-measurable $B \subseteq \mathbb{S}^d$,*
or, equivalently,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(\mathbf{x}_{k,N}) = \int_{\mathbb{S}^d} f \, \mathrm{d}\sigma_d$$

for every $f \in C(\mathbb{S}^d)$.

* Informally: A reasonable set gets a fair share of points as N becomes large.

Low-discrepancy sequences on the sphere

Spherical cap $\mathbb{L}_\infty$-discrepancy

$$D^{C}_{\mathbb{L}_\infty}(X_N) := \sup_{C} \left| \frac{|X_N \cap C|}{N} - \sigma_d(C) \right|$$

Motivated by this classical (up to $\sqrt{\log N}$ optimal) results of J. Beck, a sequence (X_N) is of **low-discrepancy** if

$$D^C_{\mathbb{L}_\infty}(X_N) \leq c_3 \frac{\sqrt{\log N}}{N^{1/2+1/(2d)}}.$$

Unresolved Question: Unlike in the unit cube case, there are no known explicit low-discrepancy constructions on the sphere.

Spherical cap $\mathbb{L}_\infty$ Discrepancy

Theorem (Aistleitner-JSB-Dick, 2012)

$$D^C_{\mathbb{L}_\infty}(Z_{F_m}) \le 44\sqrt{8}\Big/\sqrt{F_m}$$

and numerical evidence that for some $\frac{1}{2} \le c \le 1$,

$$D^C_{\mathbb{L}_\infty}(Z_{F_m}) = \mathcal{O}((\log F_m)^c\, F_m^{-3/4}) \quad \text{as } F_m \to \infty.$$

RMK: A. Lubotzky, R. Phillips and P. Sarnak (1985, 1987) have $D^C_{\mathbb{L}_\infty}(X_N^{\mathrm{LPS}}) \ll (\log N)^{2/3} N^{-1/3}$ with numerical evidence indicating $\mathcal{O}(N^{-1/2})$.

In Comparison ...

Theorem (Aistleitner-JSB-Dick, 2012)

$$\frac{c}{N^{1/2}} \leq \mathbb{E}\left[D_{\mathbb{L}_\infty}^{C}(X_N^{\text{i.i.d.}})\right] \leq \frac{C}{N^{1/2}}.$$

Surprisingly:

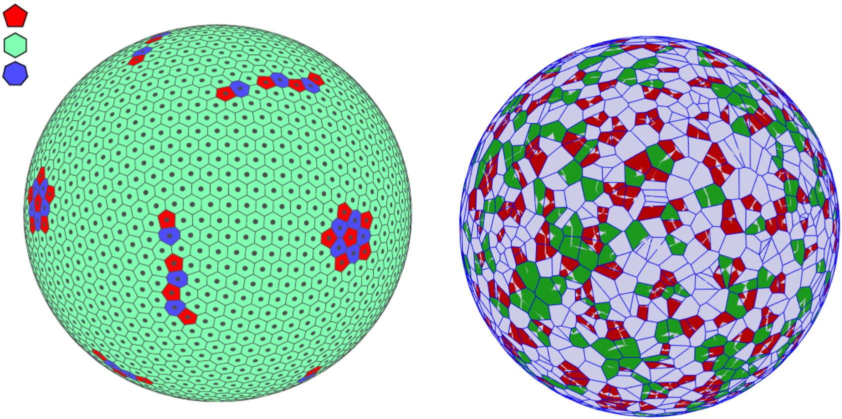
Theorem (Götz, 2000)

$$\frac{c}{N^{1/2}} \leq D_{\mathbb{L}_\infty}^{C}(X_N^{*}) \leq C\,\frac{\log N}{N^{1/2}},$$

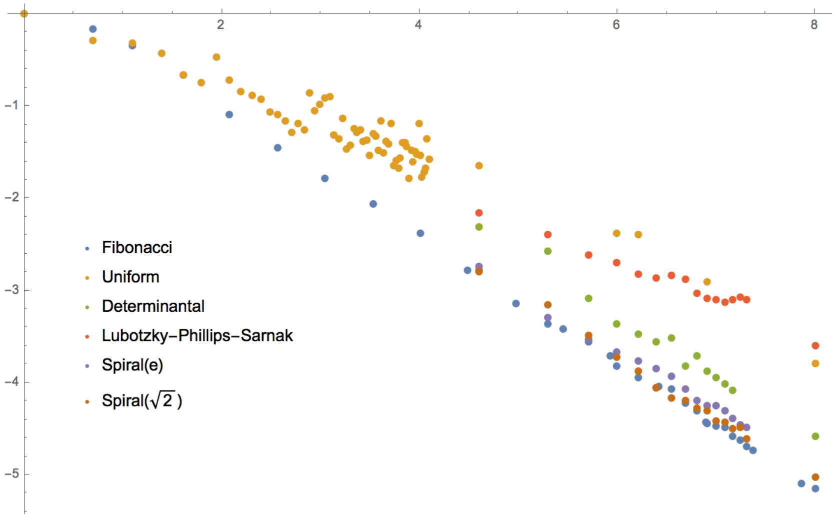
X_N^* *minimizing the* ***Coulomb potential energy***

$$\sum_{\substack{j=1 \\ j\neq k}}^{N} \sum_{k=1}^{N} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|}.$$

Examples



Near optimal Coloumb energy points (Hardin & Saff, 2004, Notices of AMS)
vs.
i.i.d. random points (courtesy of Rob Womersley)



ln-ln plot of spherical cap $\mathbb{L}_\infty$-discrepancy of point set families.

Spherical cap $\mathbb{L}_2$-discrepancy

Let $D(X_N, C) := \frac{|X_N \cap C|}{N} - \sigma_d(C)$ be the **local discrepancy function** w.r.t. spherical caps C.

The $\mathbb{L}_2$**-discrepancy** $\|D(X_N, \cdot)\|_{\mathbb{L}_2}$ satisfies

$$\frac{1}{N^2} \sum_{j,k=1}^{N} |\mathbf{x}_j - \mathbf{x}_k| + \frac{1}{c_d} \|D(X_N, \cdot)\|_{\mathbb{L}_2}^2 = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} |\mathbf{x} - \mathbf{y}| \, \mathrm{d}\,\sigma_d(\mathbf{x}) \, \mathrm{d}\,\sigma_d(\mathbf{y}),$$

an invariance principle first shown by **Stolarsky** (1973; JSB-Dick, 2013;); i.e., *maximizers of the sum of distances have optimal* $\|D(X_N, \cdot)\|_2$.†

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†The precise large N behavior is closely related to minimal Riesz energy asymptotics (JSB, 2011).

Based on results of R. Alexander, J. Beck, G. Harman, and K. Stolarsky:

Prop. (JSB, 2011)

$$\frac{c}{N^{1/2+1/(2d)}} \leq D_{\mathbb{L}_2}^{C}(X_N^{\mathrm{sum}}) \leq \frac{C}{N^{1/2+1/(2d)}}.$$

Conjecture (JSB, 2011)

$$D_{\mathbb{L}_2}^{C}(X_N^{\mathrm{sum}}) \sim \frac{A_d}{N^{1/2+1/(2d)}} \quad \text{as } N \to \infty,$$

where A_d is explicit.

Spherical Cap $\mathbb{L}_2$-discrepancy of Spherical Fibonacci Points (B–Dick, work in progress)

$$4\left[D^C_{\mathbb{L}_2}(\mathcal{Z}_n)\right]^2 = \frac{4}{3} - \frac{1}{F_n^2}\sum_{j,k=0}^{F_n-1}\left|\mathbf{z}_j - \mathbf{z}_k\right|$$

n	F_n	$4\left[D^C_{\mathbb{L}_2}(\mathcal{Z}_n)\right]^2$	$F_n^{-3/2}$	$4F_n^{3/2}\left[D^C_{\mathbb{L}_2}(\mathcal{Z}_n)\right]^2$
3	2	6.2622e-01	3.5355e-01	1.7712
4	3	3.2188e-01	1.9245e-01	1.6725
5	5	1.2865e-01	8.9442e-02	1.4384
6	8	5.7129e-02	4.4194e-02	1.2926
7	13	2.4622e-02	2.1334e-02	1.1540
8	21	1.1107e-02	1.0391e-02	1.0688
9	34	5.0965e-03	5.0440e-03	1.0103
10	55	2.3683e-03	2.4516e-03	0.9660
11	89	1.1064e-03	1.1910e-03	0.9289
12	144	5.2192e-04	5.7870e-04	0.9018
13	233	2.4792e-04	2.8116e-04	0.8817
14	377	1.1837e-04	1.3661e-04	0.8665
15	610	5.6680e-05	6.6375e-05	0.8539
16	987	2.7240e-05	3.2249e-05	0.8446
17	1597	1.3119e-05	1.5669e-05	0.8372
18	2584	6.3331e-06	7.6130e-06	0.8318
19	4181	3.0598e-06	3.6989e-06	0.8272
20	6765	1.4808e-06	1.7972e-06	0.8239
21	10946	7.1699e-07	8.7320e-07	0.8211
22	17711	3.4756e-07	4.2426e-07	0.8192
23	28657	1.6848e-07	2.0613e-07	0.8173
24	46368	8.1756e-08	1.0015e-07	0.8162
25	75025	3.9663e-08	4.8662e-08	0.8150
26	121393	1.9257e-08	2.3643e-08	0.8145
27	196418	9.3470e-09	1.1487e-08	0.8136
28	317811	4.5399e-09	5.5814e-09	0.8133
29	514229	2.2041e-09	2.7118e-09	0.8128
30	832040	1.0708e-09	1.3176e-09	0.8127
31	1346269	5.1999e-10	6.4018e-10	0.8122
				0.7985

$$\frac{1}{F_n^2}\sum_{j=0}^{F_n-1}\sum_{k=0}^{F_n-1}|\mathbf{z}_j-\mathbf{z}_k| = \frac{4}{3}-\frac{4}{3}\sum_{\ell=1}^{\infty}\frac{1}{2\ell-1}\left|\frac{1}{F_n}\sum_{k=0}^{F_n-1}P_\ell(1-\frac{2k}{F_n})\right|^2$$
$$-\frac{8}{3}\sum_{\ell=1}^{\infty}\frac{1}{2\ell-1}\sum_{m=1}^{\ell}\frac{(\ell-m)!}{(\ell+m)!}\left|\frac{1}{F_n}\sum_{k=0}^{F_n-1}P_\ell^m(1-\frac{2k}{F_n})\,e^{2\pi i\,m\,kF_{n-1}/F_n}\right|^2.$$

On the rhs one has (the error of) the numerical integration rule

$$\frac{1}{F_n}\sum_{k=0}^{F_n-1}P_\ell(1-\frac{2k}{F_n})\approx\int_{-1}^{1}P_\ell(x)\,\mathrm{d}\,x=0,\qquad \ell\geq 1,$$

with equally spaced nodes in $[-1,1]$ for Legendre polynomials P_ℓ and the *Fibonacci lattice rule*

$$\frac{1}{F_n}\sum_{k=0}^{F_n-1}P_\ell^m(1-\frac{2k}{F_n})\,e^{2\pi i\,m\,kF_{n-1}/F_n}\approx\int_0^1\int_0^1 P_\ell^m(1-2x)\,e^{2\pi i\,m\,y}\,\mathrm{d}\,x\,\mathrm{d}\,y=0$$

for Fibonacci lattice points in the unit square $[0,1]^2$ for functions

$$f_\ell^m(x,y):=P_\ell^m(1-2x)\,e^{2\pi i\,m\,y},\qquad \ell\geq 1,1\leq|m|\leq\ell.$$

PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 9 · 2016 · Pages 16–49

A Comparison of Popular Point Configurations on $\mathbb{S}^2$

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Abstract

There are many ways to generate a set of nodes on the sphere for use in a variety of problems in numerical analysis. We present a survey of quickly generated point sets on $\mathbb{S}^2$, examine their equidistribution properties, separation, covering, and mesh ratio constants and present a new point set, equal area icosahedral points, with low mesh ratio. We analyze numerically the leading order asymptotics for the Riesz and logarithmic potential energy for these configurations with total points $N < 50,000$ and present some new conjectures.

HYPERUNIFORMITY

A Bird's-Eye View of Nature's Hidden Order, Natalie Wolchover, July 12, 2016
Olena Shmahalo/Quanta Magazine; Photography: MTSOfan and Matthew Toomey
https://www.quantamagazine.org/20160712-hyperuniformity-found-in-birds-math-and-physics

▸ 2014

THE NON-COMPACT SETTING

Torquato and Stillinger [Physical Review E 68 (2003), no. 4, 041113]:

"A **hyperuniform** many-particle system in d-dimensional Euclidean space is one in which normalized density fluctuations are completely suppressed at very large lengths scales."

The *structure factor*

$$S(\mathbf{k}) = \lim_{B \to \mathbb{R}^d} \frac{1}{\#(B \cap X)} \sum_{\mathbf{x},\mathbf{y} \in B \cap X} e^{i\langle \mathbf{k}, \mathbf{x}-\mathbf{y} \rangle}$$

(thermodynamic limit)

tends to zero as $k \equiv |\mathbf{k}| \to 0$. When

$$S(\mathbf{k}) \sim |\mathbf{k}|^{\alpha} \qquad \text{as } |\mathbf{k}| \to 0,$$

where $\alpha > 0$, more can be said.

Structure Factor [a]

[a]proportional to the scattered intensity of radiation from a system of points and thus is obtainable from a scattering experiment

Scattering pattern for a **crystal** vs **disordered “stealthy” hyperuniform material**. — J. Phys.: Condens. Matter 28 (2016) 414012.

Equivalently, a

hyperuniform many-particle system

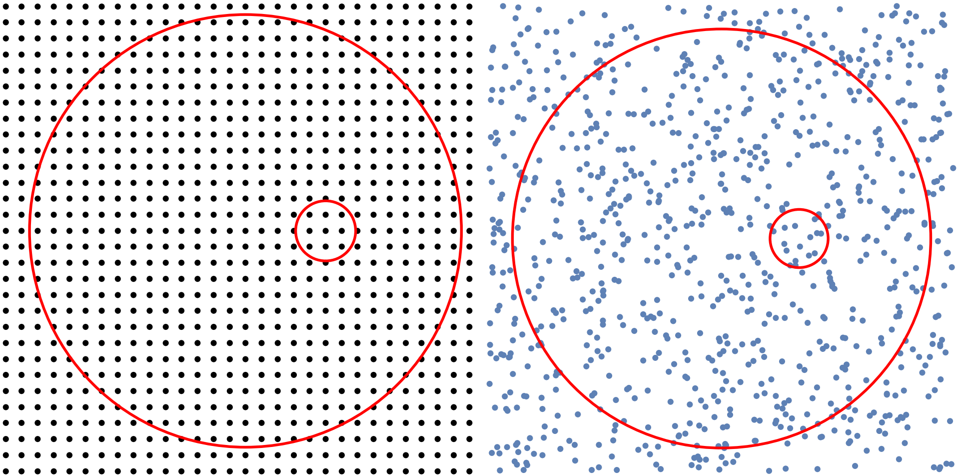
is one in which the

number variance $\mathrm{Var}[N_R]$ of particles

within a spherical observation window of radius R grows more slowly than the window volume in the large-R limit; i.e.,

slower than R^d.

Tossing observation windows



THE COMPACT SETTING

HYPERUNIFORMITY ON THE SPHERE

Constructive Approximation

pp 1–17 | Cite as

Hyperuniform Point Sets on the Sphere: Deterministic Aspects

Authors   Authors and affiliations

Johann S. Brauchart, Peter J. Grabner, Wöden Kusner

Article

First Online: 08 May 2018

https://doi.org/10.1007/s00365-018-9432-8

Infinite sequence of *N*-point sets

$$(X_N)_{N \in A}, \qquad X_N \subseteq \mathbb{S}^d, N \in A \subseteq \mathbb{N}.$$

Spherical caps

$$C(\mathbf{x}, \phi) := \left\{ \mathbf{y} \in \mathbb{S}^d \;\middle|\; \langle \mathbf{y}, \mathbf{x} \rangle > \cos(\phi) \right\}.$$

Asymptotic behavior of **number variance**

$$V(X_N, \phi) := \mathbb{V}_{\mathbf{x}}\left[\#\left(X_N \cap C(\mathbf{x}, \phi)\right)\right].$$

$$V(X_N,\phi) = \int_{\mathbb{S}^d} \Big(\sum_{n=1}^{N} \mathbb{1}_{C(\mathbf{x},\phi)}(\mathbf{x}_n) - N\,\sigma(C(\cdot,\phi)) \Big)^2 \,\mathrm{d}\,\sigma_d(\mathbf{x})$$

appears in classical measure of **uniform distribution**: **spherical cap $\mathbb{L}_2$-discrepancy**

$$D_{\mathbb{L}_2}^C(X_N) := \left(\int_0^\pi V(X_N,\phi)\,\sin(\phi)\,\mathrm{d}\,\phi \right)^{1/2},$$

where **a.u.d.** is equivalent to

$$\lim_{N\to\infty} \frac{D_{\mathbb{L}_2}^C(X_N)}{N} = 0.$$

Heuristically, **hyperuniformity in the compact setting** should mean that the number variance $V(X_N, \phi_N)$ is of

lower order than in the i.i.d. case.

Number Variance for i.i.d. points on $\mathbb{S}^d$:

$$N\,\sigma_d(C(\cdot,\phi))\Big(1-\sigma_d(C(\cdot,\phi))\Big)$$

which has order of magnitude

- **large caps:** N
- **small caps:** $N\,\sigma_d(C(\cdot,\phi_N))$

- **threshold order:** t^d if $\phi_N = t\,N^{-1/d}$

$(X_N)_{N \in A}$ is

hyperuniform for large caps if

$$V(X_N, \phi) = o(N) \qquad \text{as } N \to \infty$$

for all $\phi \in (0, \frac{\pi}{2})$.

$(X_N)_{N \in A}$ is

hyperuniform for small caps if

$$V(X_N, \phi_N) = o(N \, \sigma_d(C(\cdot, \phi_N)))$$

as $N \to \infty$ for all sequences $(\phi_N)_{N \in A}$ s.t.

$$(1) \qquad \lim_{N \to \infty} \phi_N = 0,$$

$$(2) \qquad \lim_{N \to \infty} N \, \underbrace{\sigma_d(C(\cdot, \phi_N))}_{\asymp \phi_N^d} = \infty.$$

$(X_N)_{N \in A}$ is

hyperuniform for caps
at threshold order[‡] if

$$\limsup_{N \to \infty} V(X_N, t\, N^{-\frac{1}{d}}) = \mathcal{O}(t^{d-1})$$

as $t \to \infty$.

[‡] analogous to non-compact Euclidean case

Number Variance: Technical Aspects.

Skip

$$\mathbb{1}_{C(\mathbf{x},\phi)}(\mathbf{y}) = \sigma_d(C(\mathbf{x},\phi)) + \sum_{n=1}^{\infty} a_n(\phi) \underbrace{Z(d,n)\, P_n^{(d)}(\langle \mathbf{x}, \mathbf{y} \rangle)}_{\frac{n+\lambda}{\lambda} \mathrm{C}_n^{(\lambda)}(\langle \mathbf{x},\mathbf{y}\rangle),\ \lambda=\frac{d-1}{2}},$$

where for $n \geq 1$,

$$a_n(\phi) = \frac{\gamma_d}{d} \sin(\phi)^d P_{n-1}^{(d+2)}(\cos(\phi)).$$

Laplace-Fourier expansion of $V(X_N, \phi)$

$$V(X_N, \phi) = \int_{\mathbb{S}^d} \left(\sum_{j=1}^{N} \mathbb{1}_{C(\mathbf{x}_j,\phi)}(\mathbf{x}) - N\,\sigma_d(C(\cdot,\phi)) \right)^2 \mathrm{d}\,\sigma_d(\mathbf{x}) = \sum_{i,j=1}^{N} \underbrace{\sum_{n=1}^{\infty} a_n(\phi)^2\, Z(d,n)\, P_n^{(d)}(\langle \mathbf{x}_i, \mathbf{x}_j \rangle)}_{g_\phi(\langle \mathbf{x}_i,\mathbf{x}_j \rangle)},$$

where

$$a_n(\phi)^2 = \mathcal{O}\left(\frac{\sin(\phi)^{d-1}}{n^{d+1}} \right), \quad Z(d,n) = \mathcal{O}(n^{d-1}).$$

Theorem

If $(X_N)_{N\in\mathbb{N}}$ hyperuniform for large caps, then for every $n \geq 1$

$$s(n) := \lim_{N\to\infty} \frac{1}{N} \sum_{i,j=1}^{N} P_n^{(d)}(\langle \mathbf{x}_i, \mathbf{x}_j \rangle) = 0.$$

Proof:

$$0 = \lim_{N\to\infty} \frac{V(X_N, \phi)}{N}$$

$$\geq \underbrace{a_n(\phi)^2}_{>0} \limsup_{N\to\infty} \frac{1}{N} \sum_{i,j=1}^{N} Z(d,n)\, P_n^{(d)}(\langle \mathbf{x}_i, \mathbf{x}_j \rangle).$$

$(X_N)_{N\in\mathbb{N}}$ **uniformly distributed**

- iff $\dfrac{\#(X_N \cap C)}{N} \to \sigma_d(C)$ as $N \to \infty$ for all caps C
- iff $\dfrac{1}{N^2} \displaystyle\sum_{i,j=1}^{N} Z(d,n)\, P_n^{(d)}(\langle \mathbf{x}_i, \mathbf{x}_j \rangle) \to 0$ as $N \to \infty$ for all $n \geq 1$ (Weyl criterion).

Corollary

Hyperuniform $(X_N)_{N\in\mathbb{N}}$ uniformly distributed.

NOT obvious in the small caps and threshold order regimes!

HYPERUNIFORMITY OF QMC DESIGN SEQUENCES

QMC design sequences for $\mathbb{H}^s(\mathbb{S}^d)$, $s > d/2$

Definition (JSB-Saff-Sloan-Womersley, 2014)

(X_N) is a **QMC design sequence for** $\mathbb{H}^s(\mathbb{S}^d)$ if there is a $c(s,d) > 0$ independent of N s.t.

$$\left| \frac{1}{N} \sum_{\mathbf{x} \in X_N} f(\mathbf{x}) - \int_{\mathbb{S}^d} f \, \mathrm{d}\sigma_d \right| \leq \frac{c(s,d)}{N^{s/d}} \|f\|_{\mathbb{H}^s}$$

for $f \in \mathbb{H}^s(\mathbb{S}^d)$.

$c(s,d)$ may depend on $\mathbb{H}^s(\mathbb{S}^d)$-norm and (X_N).

Theorem

A QMC design sequence for $\mathbb{H}^s(\mathbb{S}^d)$ with $s \geq \frac{d+1}{2}$ is hyperuniform for large caps, small caps, and caps at threshold order.

Lemma

The number variance satisfies

$$V(X_N, \phi) \ll (\sin \phi)^{d-1} N^2 \left[\mathrm{wce}(Q[X_N]; \mathbb{H}^{\frac{d+1}{2}}(\mathbb{S}^d))\right]^2$$

for any N-point set $X_N \subseteq \mathbb{S}^d$ and opening angle $\phi \in (0, \frac{\pi}{2})$.

A sequence $(Z^*_{N_t})$ of spherical t-designs with N_t points of exactly the optimal order ($N_t \asymp t^d$) of points has the remarkable property that

$$\left| \mathrm{Q}[Z_{N_t^*}](f) - \mathrm{I}(f) \right| \leq \frac{c^*(s,d)}{N_t^{s/d}} \, \|f\|_{\mathbb{H}^s}$$

for all $f \in \mathbb{H}^s(\mathbb{S}^d)$ and **all** $s > \frac{d}{2}$.

The order of N_t cannot be improved.

Example: Maximal Sum-of-Distances Points

A sequence (X_N^{sum}) of maximal sum-of-distance N-point sets define QMC rules that satisfy

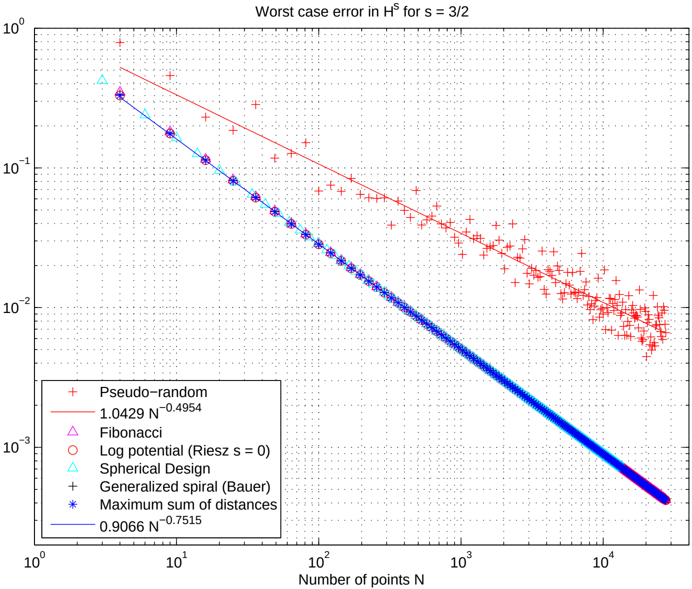
$$\left|\mathrm{Q}[X_N^{\mathrm{sum}}](f) - \mathrm{I}(f)\right| \leq \frac{c^{\mathrm{sum}}(s,d)}{N^{s/d}} \|f\|_{\mathbb{H}^s}$$

for all $f \in \mathbb{H}^s(\mathbb{S}^d)$ and all $\frac{d}{2} < s \leq \frac{d+1}{2}$.

The order of N cannot be improved.

Open: Determine ***strength*** of (X_N^{sum}).

Numerics



(cf. (JSB-Saff-Sloan-Womersley, 2014))

HYPERUNIFORMITY: PROBABILISTIC ASPECTS

▸ Skip

Joint work with

- Peter J. Grabner
  (Graz University of Technology)
- Wöden Kusner (Vanderbilt University)
- Jonas Ziefle (University of Tübingen)

https://arxiv.org/abs/1809.02645

Given: point process $\mathcal{X}_N$ by **joint densities**

$$(\mathbf{X}_1, \dots, \mathbf{X}_N) \sim \rho^{(N)}$$

invariant under

- permutation (exchangeable particles) and
- isometries of the sphere.

Then: Number variance

$$\begin{aligned} \mathbb{V}\mathcal{X}_N(B) &= \mathbb{E}(\mathcal{X}_N(B))^2 - (\mathbb{E}\mathcal{X}_N(B))^2 \\ &= N\sigma_d(B)\,(1 - \sigma_d(B)) \\ &\quad + N(N-1) \int_B \int_B \left(\rho_2^{(N)}(\mathbf{x}, \mathbf{x}') - 1\right) \\ &\qquad \times \mathrm{d}\,\sigma_d(\mathbf{x})\,\mathrm{d}\,\sigma_d(\mathbf{x}'). \end{aligned}$$

We study, in particular:

- Spherical ensembles on $\mathbb{S}^2$ and $\mathbb{S}^{2d}$
  hyperuniform
- Harmonic ensemble on $\mathbb{S}^d$
  hyperuniform, except treshold order
- Jittered sampling on $\mathbb{S}^d$
  hyperuniform, determinantal;

Thank You!

APPENDIX

PHYSICAL REVIEW E **89**, 022721 (2014)

Avian photoreceptor patterns represent a disordered hyperuniform solution to a multiscale packing problem

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(Received 6 November 2013; published 24 February 2014)

Optimal spatial sampling of light rigorously requires that identical photoreceptors be arranged in perfectly

Return

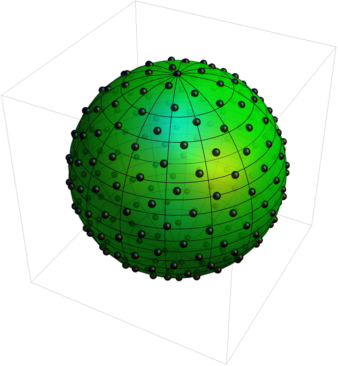
Discrete Comput Geom (2012) 48:990–1024
DOI 10.1007/s00454-012-9451-3

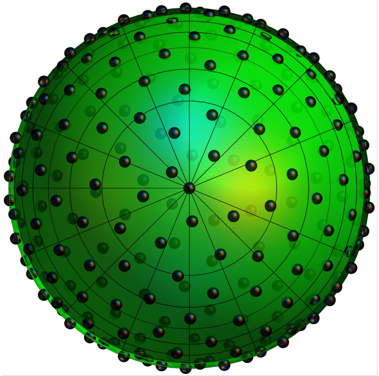
Point Sets on the Sphere $\mathbb{S}^2$ with Small Spherical Cap Discrepancy

C. Aistleitner · J.S. Brauchart · J. Dick

Received: 14 September 2011 / Revised: 31 July 2012 / Accepted: 7 August 2012 /
Published online: 28 August 2012

Spherical Fibonacci Lattice Points





Fibonacci sequence (OEIS: A000045):

$$F_0 := 0,$$
$$F_1 := 1,$$
$$F_{n+1} := F_n + F_{n-1}, \qquad n \geq 1.$$

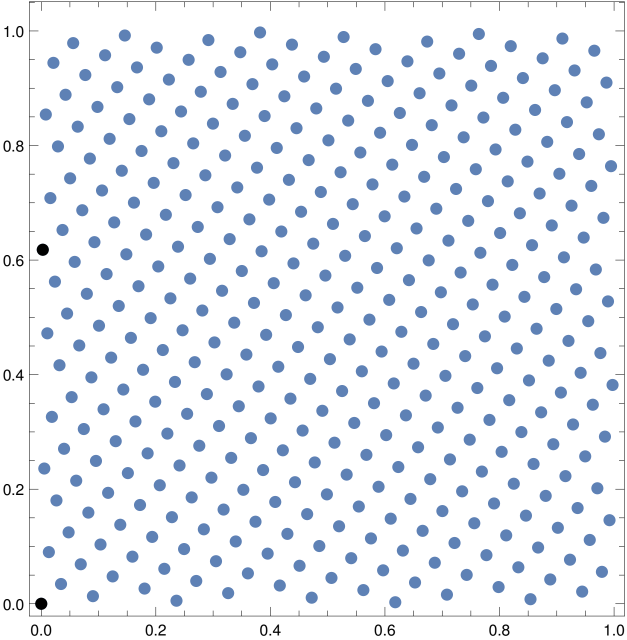
Fibonacci lattice in $[0,1]^2$

$$\mathcal{F}_n : \qquad \left(\frac{k}{F_n}, \left\{ k \frac{F_{n-1}}{F_n} \right\} \right), \quad 0 \leq k < F_n,$$

has optimal order star-discrepancy bounds:

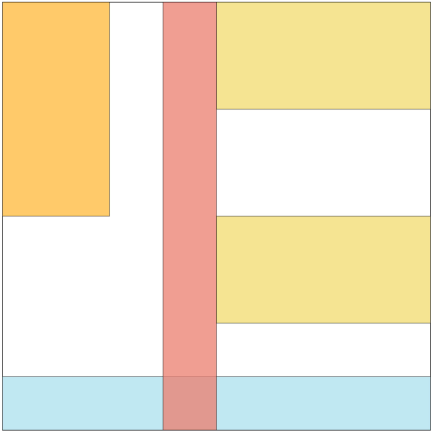
$$\|\mathcal{D}(\mathcal{F}_n; \cdot)\|_\infty \asymp n \asymp \log F_n.$$

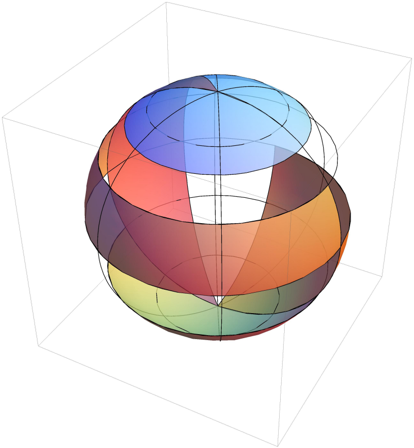
$\{x\}$ is fractional part of real x.



$n = 14$: $F_n = 377$.

Area preserving **Lambert transformation** $\mathbf{\Phi} : [0,1]^2 \to \mathbb{S}^2$





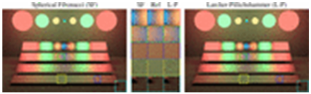
$$\mathbf{\Phi}(x,y) = \begin{pmatrix} 2\cos(2\pi y)\sqrt{x - x^2} \\ 2\sin(2\pi y)\sqrt{x - x^2} \\ 1 - 2x \end{pmatrix}$$

Return

Spherical Fibonacci Point Sets for Illumination Integrals (pages 134–143)

R. Marques, C. Bouville, M. Ribardière, L. P. Santos and K. Bouatouch

Article first published online: 24 JUL 2013 | DOI: 10.1111/cgf.12190



Quasi-Monte Carlo (QMC) methods exhibit a faster convergence rate than that of classic Monte Carlo methods. This feature has made QMC prevalent in image synthesis, where it is frequently used for approximating the value of spherical integrals (e.g. illumination integral). The common approach for generating QMC sampling patterns for spherical integration is to resort to unit square low-discrepancy sequences and map them to the hemisphere. However such an approach is suboptimal as these sequences do not account for the spherical topology and their discrepancy properties on the unit square are impaired by the spherical projection.

Estimates for s^* for $d = 2$

$$s^* := \sup \left\{ s : \begin{array}{l} (X_N) \text{ is QMC design} \\ \text{sequence for } \mathbb{H}^s(\mathbb{S}^d) \end{array} \right\}.$$

Table: Estimates of s^* for $d = 2$

Point set	s^*
Fekete	1.5
Equal area	2
Coulomb energy	2
Log energy	3
Generalized spiral	3
Distance	4
Spherical designs	∞