A KdV soliton gas: asymptotic analysis via Riemann–Hilbert problems

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The KdV equation

In 1834 the Scottish engineer John Scott-Russell accidentally observed a surface water wave in the Union Canal between Edinburgh and Glasgow that appeared to be a spatially localized traveling wave, that he called “great wave of translation”.

In 1895, D. J. Korteweg and G. de Vries proposed the following equation to describe this phenomenon:

$$u_t - 6uu_x + u_{xxx} = 0.$$

The one-soliton solution

The simplest wave solution is:

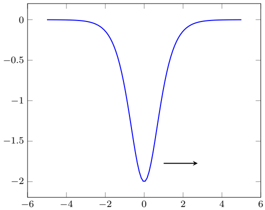
$$u(x,t) = \varphi_v\,(x - vt)\,.$$

With this ansatz, the PDE becomes an ODE in the variable $\xi = x - vt$

$$-v\varphi_v' - 6\varphi_v\varphi_v' + \varphi_v''' = 0$$

One solution is a rapidly decreasing, localized travelling wave (soliton):

$$u(x,t) = -\frac{v}{2}\,\mathrm{sech}^2\left(\frac{\sqrt{v}}{2}(x - vt - x_0)\right)$$



$$u(x,t) = -\frac{v}{2}\operatorname{sech}^2\left(\frac{\sqrt{v}}{2}(x - vt - x_0)\right)$$

Remark

- *In order to have a real solution, we need $v > 0$, which in turn implies that the wave-solution can move only to the right.*
- *The amplitude of the wave is proportional to the speed v, thus larger amplitude solitary waves move with a higher speed than smaller amplitude waves.*

The periodic soliton solution

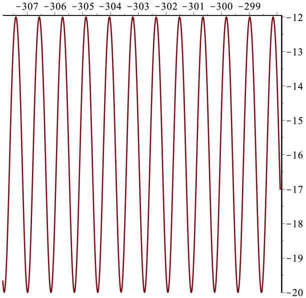
Starting again from the ansatz:

$$u(x,t) = \varphi_v\,(x - vt)$$

and imposing a periodicity condition, the solution (periodic travelling wave) can be written in terms of Jacobi elliptic functions:

$$u(x,t) = \beta_1 - \beta_2 - \beta_3 - 2(\beta_1 - \beta_3)\,\mathrm{dn}^2\left(\sqrt{\beta_1 - \beta_3}(x - 2(\beta_1 + \beta_2 + \beta_3)t) + x_0 \,|\, m\right)$$

where $\mathrm{dn}\,(z\,|\,m)$ is the Jacobi elliptic function of modulus $m = \frac{\beta_2 - \beta_3}{\beta_1 - \beta_3}$ and $\beta_1 > \beta_2 > \beta_3$.



Looking for other solutions...

The Cauchy problem (Gardner-Greene-Kruskal-Miura, '67) :

$$\begin{cases} u_t - 6uu_x + u_{xxx} = 0 \\ u(x,0) = q(x) \end{cases}$$

for rapidly decaying initial data: $q(x) \to 0$ as $x \to \pm\infty$.

This nonlinear PDE is **integrable**, arising as the compatibility condition of a Lax pair of linear differential operators (Lax, '68):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L} = [\mathcal{B}, \mathcal{L}]$$

with

$$\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u, \qquad \mathcal{B} = -4\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 6u\frac{\mathrm{d}}{\mathrm{d}x} + 3u_x \ .$$

Equivalently, the compatibility condition can be presented as the existence of a simultaneous solution to the pair of equations:

$$\mathcal{L}\phi = E\phi, \qquad \phi_t = \mathcal{B}\phi$$

where $E \in \mathbb{R}$ is the spectral parameter.

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Solving the Schrödinger equation

We start from

$$\mathcal{L}\phi = E\phi,$$

where $\mathcal{L} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ is the Schrödinger operator with potential $V(x) = u(x,0) = q(x)$ (no dependence on time... yet!).

Using tools from spectral theory, GGKM calculated the **scattering data**, which will allow to find the solution ϕ to the Schrödinger equation:

$\mathcal{S} = \{-\lambda_1^2, \dots, -\lambda_n^2$ eigenvalues,

$c_1, \dots, c_n$ norming constant of the eigenfunctions,

$r(\lambda)$ reflection coefficient of the "scattering" solutions $\phi_\pm(x)\}$

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Turning on time

If the potential $V_t(x) = u(x,t)$ depends also on a (time) parameter t, one expects the scattering data $\mathcal{S} = \left\{\{-\lambda_j^2\}, \{c_j\}, r(\lambda)\right\}$ to vary with t as well.

If the t dependence of $u(x,t)$ is given in terms of the KdV equation,

$$u_t = -u_{xxx} + 6uu_x,$$

then the scattering data $\mathcal{S}(t)$ evolve in a very simple and explicit manner (GGMK, '67):

1. the discrete eigenvalues are constant: $E = -\lambda_j^2$;
2. the norming constants have exponential behaviour: $c_j(t) = c_j(0)e^{A\lambda_j^3 t}$;
3. same for the reflection coefficient: $r(\lambda;t) = r(\lambda;0)e^{iB\lambda^3 t}$.

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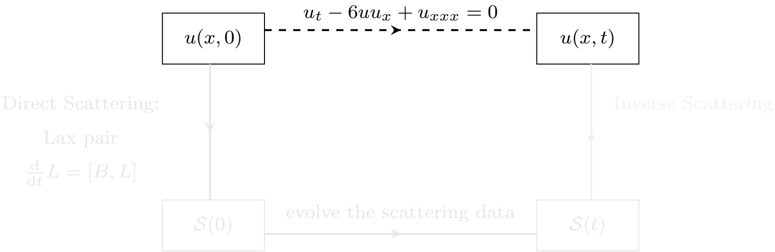
Solve the Cauchy initial-value problem for KdV

Recipe:



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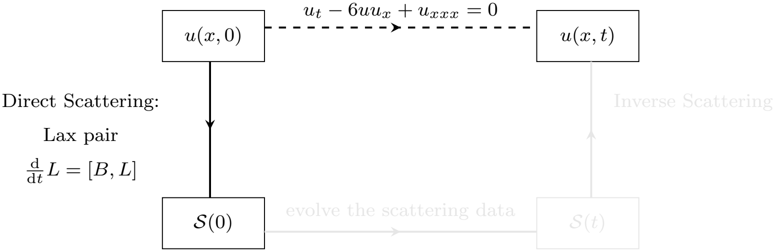

Calculate the scattering data: $\mathcal{S} = \left\{ \{-\lambda_j^2\}, \{c_j\}, r(\lambda) \right\}$

Solve the Cauchy initial-value problem for KdV

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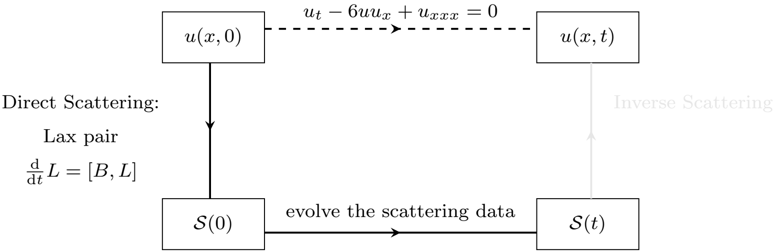

Calculate the time-evolved scattering data $\mathcal{S}(t)$, imposing $u(x,t)$ to be a solution of KdV: $u_t = 6uu_x - u_{xxx}$.

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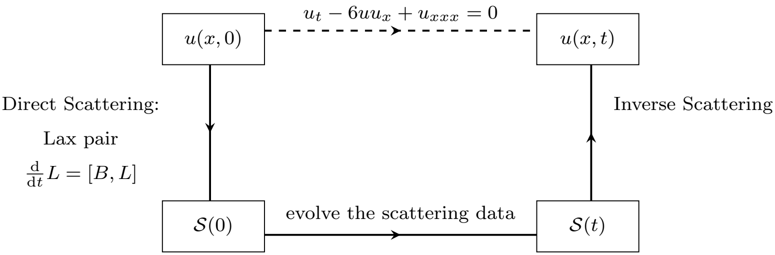
Recipe:



Construct the inverse scattering map to obtain the solution $u(x,t)$:

- Marchenko integral equation (Gelfand-Levitan-Marchenko, 1950's)
- Riemann–Hilbert problem (Deift-Zhou, '93; Grunert–Teschl, '09)

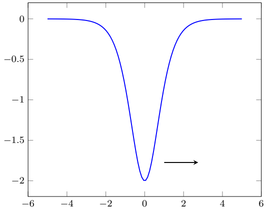
Where is the soliton?

Suppose that $u(x,0) = q(x)$ is such that the corresponding Schrödinger operator has only one eigenvalue $E = -\lambda_1^2$ and no reflection coefficient $r(\lambda) \equiv 0$.

The solution $u(x,t)$ is a 1-soliton solution:

$$u(x,t) = -\frac{v}{2}\,\mathrm{sech}^2\left(\frac{\sqrt{v}}{2}(x - vt - x_0)\right)$$

where $v = \lambda_1^2$.



In general,

1. (Multi)-soliton solutions correspond to the (discrete) eigenvalues $\{-\lambda_j^2\}$ of the Schrödinger operator $\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u$.
2. The reflection coefficient $r(\lambda)$ corresponds to a radiative part (associated to the continuous spectrum). Qualitatively, the linear radiation propagates to the left and the amplitude decays in time at rate t^{-1}.

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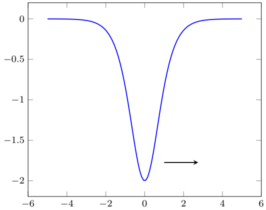

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What's so special about solitons?

- Solitons are solitary wave (localised travelling wave) solution of the KdV equation.
- Solitons corresponds to the (discrete) eigenvalues of the Schrödinger operator and they arise in the long-time behaviour of the solution.
- The interaction between solitons is elastic!

$$u(x,t) \to \sum_{j=1}^{N} \varphi_{v_j}\left(x - v_j t + \delta_j^{\pm}\right) \qquad \text{as } t \to \pm\infty$$

They "survive" collisions (Zabusky-Kruskal, '65), despite lack of superposition principle.

(courtesy of Peter Miller)

1. Background and motivations
   - KdV and solitons
   - The soliton gas and the Riemann–Hilbert problem
2. Asymptotics of the initial condition $u(x, 0)$ for large x's
3. Large time behaviour of the potential $u(x, t)$
   - The super-critical case
   - The sub-critical case
4. To be continued...

What is a soliton gas? (Zakharov, '71)

Recent interest revolves around the computation of statistical quantities describing the evolution of random configurations of a large number of solitons ("soliton ensemble").

Let

$$f_\lambda(x,t)\,\mathrm{d}\lambda\,\mathrm{d}x = \left\{ \begin{array}{c} \text{number of solitons with the spectral parameter } (\lambda, \lambda+\mathrm{d}\lambda) \\ \text{located in the spatial interval } (x, x+\mathrm{d}x) \text{ at time } t \end{array} \right\}$$

Definition

A **soliton gas** is an infinite collection of solitons randomly distributed on $\mathbb{R}$ with non-zero (physical) density

$$\varrho(x,t) = \int_I f_\lambda(x,t)\,\mathrm{d}\lambda.$$

The nonlinear wave field

$$u(x,t)$$

solving the KdV equation in this setting is called **integrable soliton turbulence**.

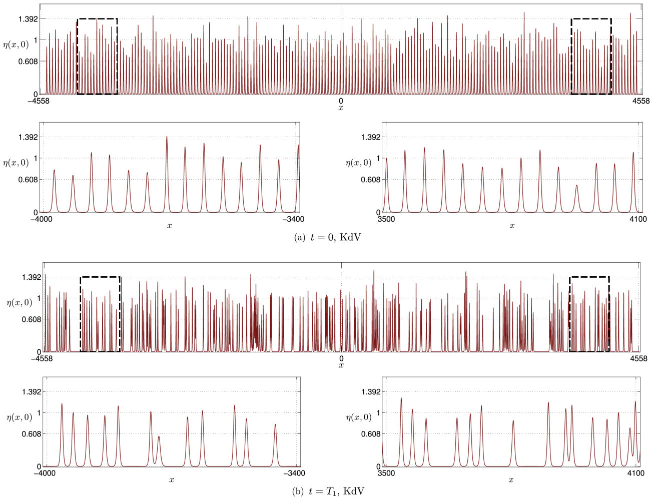
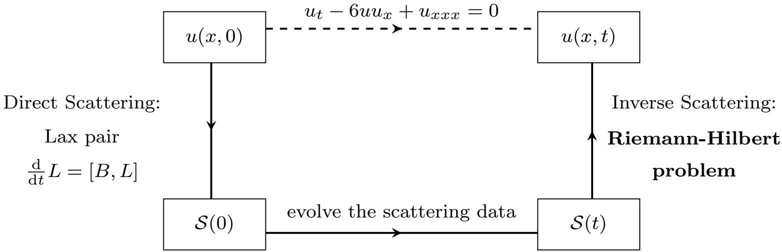


Figure: The initial condition (a) and the final state (b) of a random KdV soliton gas simulated with $N = 200$ solitons. From *Dutykh, Pelinovsky, '14*.

Find the solution of a KdV soliton gas equation

Recipe:



What is a RH problem...

RH problem

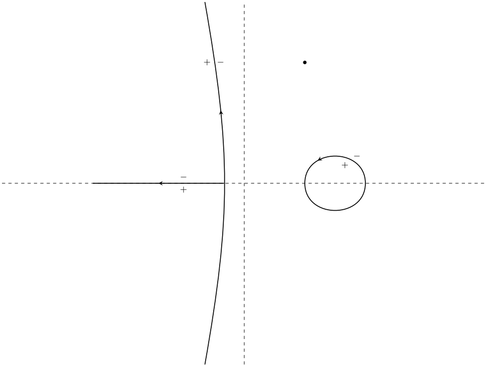
Given a set of oriented contours Σ in the complex plane, find a (matrix-valued) function X such that:

1. *X is holomorphic in $\mathbb{C} \setminus \Sigma$;*
2. *jump condition: there exists (finite) the limit of X as λ approaches the contours $X_\pm(\lambda)$ such that*

$$X_+(\lambda) = X_-(\lambda) J(\lambda) \qquad \lambda \in \Sigma;$$

3. *normalization at infinity:*

$$X(\lambda) = I + \mathcal{O}\left(\frac{1}{\lambda}\right) \qquad \lambda \to \infty.$$



Remark

Explicit solutions are extremely rare!

Soliton gas as a limit of N solitons

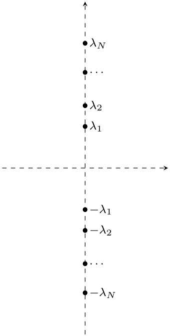
A pure N-soliton solution ($r(\lambda) \equiv 0$) is described by

$M(\lambda) \in \mathrm{Vec}_2(\mathbb{C})$ meromorphic in $\mathbb{C} \setminus \left\{\lambda_j, \overline{\lambda}_j\right\}_{j=1}^{N}$

$$\operatorname*{res}_{\lambda=\lambda_j} M = \lim_{\lambda\to\lambda_j} M(\lambda) \begin{bmatrix} 0 & 0 \\ \frac{c_j e^{2i\lambda_j x}}{N} & 0 \end{bmatrix}$$

$$\operatorname*{res}_{\lambda=\overline{\lambda_j}} M = \lim_{\lambda\to\overline{\lambda_j}} M(\lambda) \begin{bmatrix} 0 & \frac{-c_j e^{-2i\overline{\lambda_j} x}}{N} \\ 0 & 0 \end{bmatrix}$$

$$M(\lambda) = \begin{bmatrix} 1 & 1 \end{bmatrix} + \mathcal{O}\left(\lambda^{-1}\right) \qquad \lambda \to \infty$$



with

$$c_j = \frac{i(\eta_2 - \eta_1)}{\pi} r_1(\lambda_j).$$

And the solution u can be recovered as

$$u(x) = 2\frac{\mathrm{d}}{\mathrm{d}x}\left[\lim_{\lambda\to\infty} \frac{\lambda}{i}\left(M_1(\lambda) - 1\right)\right].$$

Then, we take the limit as $N \nearrow +\infty$ assuming that the poles/solitons accumulates within $[i\eta_1, i\eta_2] \cup [-i\eta_2, -i\eta_1]$ and we obtain the RH problem for a soliton gas.

Theorem (G., Grava, McLaughlin, '18)

The Riemann-Hilbert problem for a KdV soliton gas can be derived as a (uniform) limit of a meromorphic Riemann-Hilbert problem for N solitons as $N \nearrow +\infty$.

$X(\lambda) \in \mathrm{Vec}_2(\mathbb{C})$ meromorphic in $\mathbb{C} \setminus \{i\Sigma_1 \cup i\Sigma_2\}$

$$X_+(\lambda) = X_-(\lambda) \begin{cases} \begin{bmatrix} 1 & 0 \\ -i\,r_1(\lambda)\,e^{2i\lambda x} & 1 \end{bmatrix} & \lambda \in i\Sigma_1 \\ \begin{bmatrix} 1 & i\,r_1(\lambda)\,e^{-2i\lambda x} \\ 0 & 1 \end{bmatrix} & \lambda \in i\Sigma_2 \end{cases}$$

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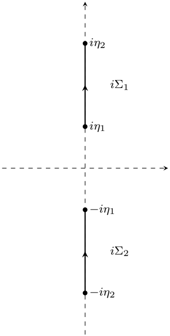

Finally, the solution u is still given as

$$u(x) = 2\frac{\mathrm{d}}{\mathrm{d}x}\left[\lim_{\lambda\to\infty}\frac{\lambda}{i}\left(X_1(\lambda)-1\right)\right].$$

Remark

This RH problem is a special case of the soliton gas RH problem proposed by Dyachenko-Zakharov-Zakharov ('16).

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The RH problem for the potential at initial time

$$
Y_+(\lambda) = Y_-(\lambda) \begin{cases} \begin{bmatrix} 1 & 0 \\ -i\, r(\lambda)\, e^{-2\lambda x} & 1 \end{bmatrix} & \lambda \in [\eta_1, \eta_2] =: \Sigma_1 \\ \begin{bmatrix} 1 & i\, r(\lambda)\, e^{2\lambda x} \\ 0 & 1 \end{bmatrix} & \lambda \in [-\eta_2, -\eta_1] =: \Sigma_2 \end{cases}
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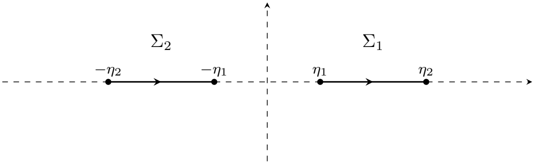

We recover $u(x)$ as

$$
u(x) = 2 \frac{\mathrm{d}}{\mathrm{d}x} \left[\lim_{\lambda \to \infty} \lambda \left(Y_1(\lambda) - 1 \right) \right].
$$

Large positive x's

When $x \nearrow +\infty$, we have

$$e^{-2\lambda x} \to 0 \quad \text{on } \Sigma_1 \qquad \text{and} \qquad e^{2\lambda x} \to 0 \quad \text{on } \Sigma_2,$$

leaving us with

$$Y_+(\lambda; x) = Y_-(\lambda; x) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \lambda \in \Sigma_1 \cup \Sigma_2$$
$$Y(\lambda; x) = \begin{bmatrix} 1 & 1 \end{bmatrix} + \mathcal{O}\left(\frac{1}{\lambda}\right) \qquad \lambda \to \infty$$

up to exponentially small terms.

Then the solution is clearly

$$Y = \begin{bmatrix} 1 & 1 \end{bmatrix} + \{\text{small terms}\}$$

and the KdV potential is

$$u(x) = 2\frac{\mathrm{d}}{\mathrm{d}x}\left[\lim_{\lambda\to\infty} \lambda\left(Y_1 - 1\right)\right] = 0 + \{\text{small terms}\} \qquad \text{for } x \gg 1.$$

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Large negative x's: the Deift–Zhou steepest descent method

On the other hand, when $x \searrow -\infty$, we have

$$e^{\mp 2\lambda x} \to +\infty \quad \text{on } \Sigma_{1/2}.$$

Steepest Descent Method (Deift-Zhou, '93): the strategy is to perform a sequence of (invertible) transformations of the original RH problem Y

$$Y \mapsto T \mapsto U \mapsto \ldots \mapsto S$$

in such away that, in the regime $x \ll -1$, the final RH problem S can be solved by an approximating solution Ω (the "model problem"):

$$S \sim \Omega.$$

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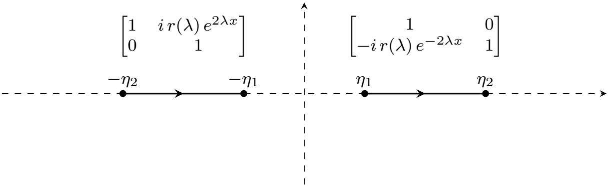

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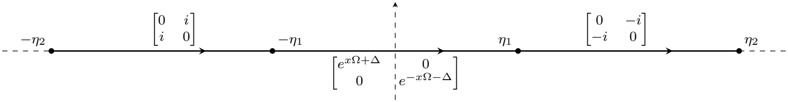
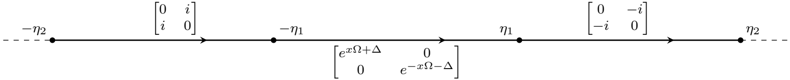


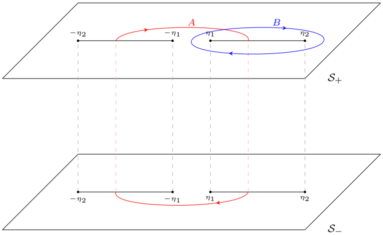
The (matrix) model problem

The global parametrix $P^{(\infty)}$: $P_+^{(\infty)}(\lambda) = P_-^{(\infty)}(\lambda) J_\infty$



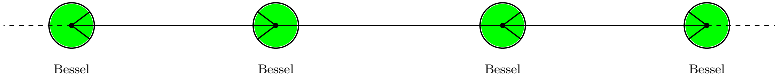
with $P^{(\infty)}(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mathcal{O}\left(\lambda^{-1}\right)$ as $\lambda \to \infty$.

The construction of the solution relies on the ϑ-function associated to the genus-1 Riemann surface $\mathfrak{X} = \left\{(\lambda, \eta) \in \mathbb{C}^2 \mid \eta^2 = (\lambda^2 - \eta_1^2)(\lambda^2 - \eta_2^2)\right\}$.



$P^{(\infty)}$ is a good approximation of S everywhere on $\mathbb{C}$ except at the endpoints $\lambda = \pm\eta_2, \pm\eta_1$, where it exhibits a fourth-root singularity, while S is bounded in a neighbourhood of those points.

Four local (matrix) parametrices $P^{(\pm\eta_j)}$:



Call Ω the "alleged" approximant built out of the global parametrix $P^{(\infty)}$ and the four local parametrices $P^{(\pm\eta_j)}$.

Question: how well does Ω approximate S?

$$S(\lambda) \sim \Omega(\lambda).$$

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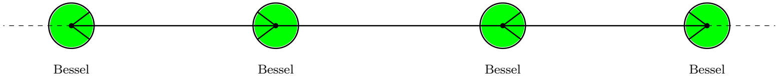

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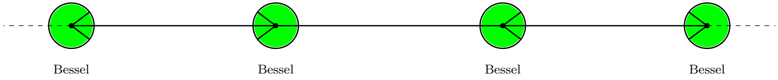

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Small-norm argument

Consider the ratio

$$R := S\Omega^{-1}.$$

Then,

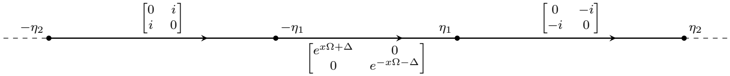
$$\begin{cases} R_+(\lambda) = R_-(\lambda)\left(I + \delta V(\lambda)\right) & \text{on the contours, with } \delta V = \mathcal{O}\left(|x|^{-*}\right) \\ R(\lambda) = \begin{bmatrix} 1 & 1 \end{bmatrix} + \mathcal{O}\left(\dfrac{1}{\lambda}\right) & \lambda \to \infty. \end{cases}$$

It follows that

$$R(\lambda) = \begin{bmatrix} 1 & 1 \end{bmatrix} + \mathcal{O}\left(|x|^{-*}\right),$$

meaning

$$S(\lambda) = R(\lambda)\Omega(\lambda) = \left(\begin{bmatrix} 1 & 1 \end{bmatrix} + \mathcal{O}\left(|x|^{-*}\right)\right)\Omega(\lambda)$$



The solution

Theorem (G., Grava, McLaughlin, '18)

In the regime $x \searrow -\infty$, with $\frac{x\Omega+\Delta}{2\pi i} \neq \frac{2n+1}{2}$, $n \in \mathbb{Z}$, the potential $u(x)$ has the following asymptotic behaviour

$$u(x) = \eta_2^2 - \eta_1^2 - 2\eta_2^2 \,\mathrm{dn}^2\left(\eta_2(x+\phi) + K(m) \,|\, m\right) + \mathcal{O}\left(|x|^{-1}\right)$$

where $\mathrm{dn}\,(z\,|\,m)$ is the Jacobi elliptic function of modulus $m = \eta_1/\eta_2$, $K(m)$ is the complete elliptic integrals of second kind of modulus m and ϕ is given by

$$\phi = \int_{\eta_1}^{\eta_2} \frac{\log r(\zeta)}{R_+(\zeta)} \frac{\mathrm{d}\zeta}{\pi i} \in \mathbb{R}\,.$$

$u(x)$ is a periodic wave with

- period $= \dfrac{2K(m)}{\eta_2}$;
- amplitude $= 2\eta_1^2$;
- average value of $u(x)$ over an oscillation:

$$< u(x) >= \eta_2^2 - \eta_1^2 - 2\eta_2^2 \frac{E(m)}{K(m)}\,.$$

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Note:
There is an issue for some values of x: for

$$\frac{x\Omega + \Delta}{2\pi i} = \frac{2n+1}{2}, \quad n \in \mathbb{Z},$$

we cannot build a matrix model problem, therefore the small norm argument cannot be used.

Work in progress...

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   - The sub-critical case

4. To be continued...

Switching on time!

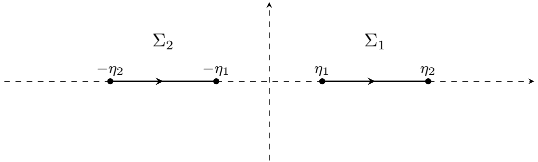
Replace

$$2\lambda x \mapsto 2\lambda x - 8\lambda^3 t$$

in the exponentials (evolution of the reflection coefficient).

$$Y_+(\lambda) = Y_-(\lambda) \begin{cases} \begin{bmatrix} 1 & 0 \\ -i\, r(\lambda)\, e^{-2\lambda x + 8\lambda^3 t} & 1 \end{bmatrix} & \lambda \in \Sigma_1 \\ \\ \begin{bmatrix} 1 & i\, r(\lambda)\, e^{2\lambda x - 8\lambda^3 t} \\ 0 & 1 \end{bmatrix} & \lambda \in \Sigma_2 \end{cases}$$

$$Y(\lambda) = \begin{bmatrix} 1 & 1 \end{bmatrix} + \mathcal{O}\left(\frac{1}{\lambda}\right) \qquad \lambda \to \infty.$$



The phase in the jumps

$$2\lambda x - 8\lambda^3 t = -8t\lambda\left(\lambda^2 - \frac{x}{4t}\right)$$

shows different sign depending on the value of the quantity

$$\boxed{\xi := \frac{x}{4t}}$$

There are three main domains:

- $\eta_2^2 < \xi$ (**trivial case**): the phases are exponentially decaying as $t \nearrow +\infty$, therefore (by a small norm argument)

$$u(x,t) = \mathcal{O}\left(t^{-\infty}\right).$$

- $\xi_{\text{crit}} < \xi < \eta_2^2$ (**super-critical case**): $u(x,t)$ is a periodic travelling wave with slowly varying parameters.
- $\xi < \xi_{\text{crit}}$ (**sub-critical case**): $u(x,t)$ is a periodic travelling wave with fixed parameters.

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The super-critical case: α-dependency

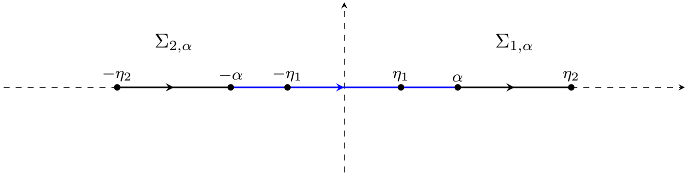
Proposition

Let $\xi < \eta_2^2$. There exists

$$\xi_{\text{crit}} \in \mathbb{R}$$

such that

$$\textit{for each } \xi \in [\xi_{\text{crit}}, \eta_2^2] \textit{ there exists a unique } \alpha = \alpha(\xi; \eta_1, \eta_2) \in [\eta_1, \eta_2]\,.$$



We can now proceed with the transformations

$$Y \overset{g-\text{function}}{\longrightarrow} T \overset{\text{opening lenses}}{\longrightarrow} S$$

and get to the model problem $\Omega(\lambda)$.

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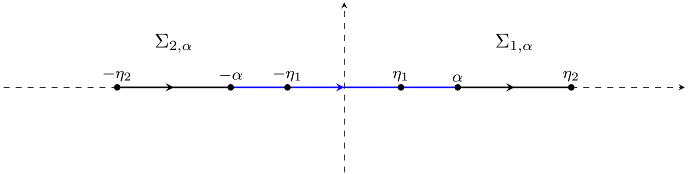

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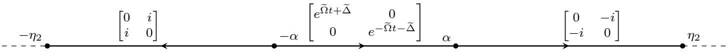
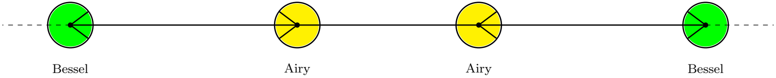
The (matrix) model problem

The global parametrix $P^{(\infty)}$:



the construction of the solution relies on the ϑ_3-function associated to the genus-1 Riemann surface $\mathfrak{X}_\alpha = \left\{(\lambda, \eta) \in \mathbb{C}^2 \mid \eta^2 = R_\alpha^2(\lambda) = (\lambda^2 - \alpha^2)(\lambda^2 - \eta_2^2)\right\}$.

Plus four local parametrices $P^{(\pm\eta_2)}$ and $P^{(\pm\alpha)}$:



Back to the potential $u(x,t)$

Theorem (G., Grava, McLaughlin, '18)

Given $\xi = \frac{x}{4t}$, *for* $\frac{t\widetilde{\Omega}+\widetilde{\Delta}}{2\pi i} \neq \frac{2n+1}{2}$, $n \in \mathbb{Z}$, *in the region* $\xi_{\text{crit}} < \xi < \eta_2^2$ *the solution of the KdV equation in the large time limit is*

$$u(x,t) = \eta_2^2 - \alpha^2 - 2\eta_2^2 \,\mathrm{dn}^2\left(\eta_2(x - 2(\alpha^2+\eta_2^2)t + \widetilde{\phi}) + K(m_\alpha) \,|\, m_\alpha\right) + \mathcal{O}\left(t^{-1}\right)$$

where $\mathrm{dn}\,(z\,|\,m)$ *is the Jacobi elliptic function of modulus* $m_\alpha = \frac{\alpha}{\eta_2}$,

$$\widetilde{\phi} = \int_\alpha^{\eta_2} \frac{\log r(\zeta)}{R_{\alpha+}(\zeta)} \frac{\mathrm{d}\zeta}{\pi i} \in \mathbb{R}$$

and the parameter $\alpha = \alpha(\xi)$ *is determined from the equation*

$$\xi = \frac{\eta_2^2}{2}\left[1 + m_\alpha^2 + 2\frac{m_\alpha^2(1-m_\alpha^2)}{1 - m_\alpha^2 - \frac{E(m_\alpha)}{K(m_\alpha)}}\right].$$

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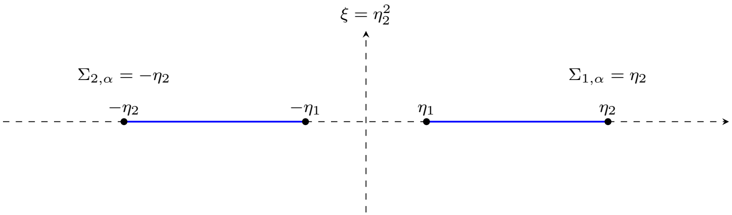
The sub-critical case

For $\xi < \xi_{\text{crit}}$ we have a phase transition:



Proposition

The value of α monotonically decreases as ξ decreases for $\xi \in [\xi_{\text{crit}}, \eta_2^2]$.

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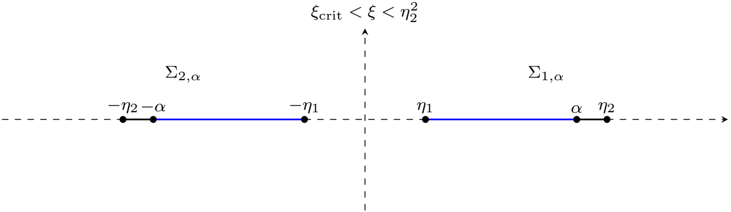

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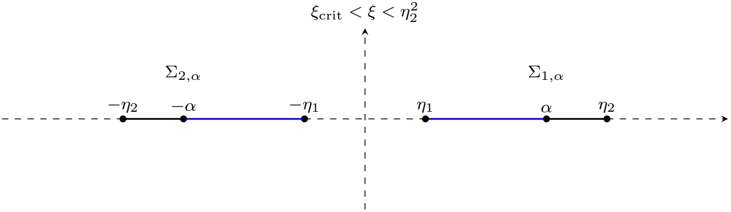

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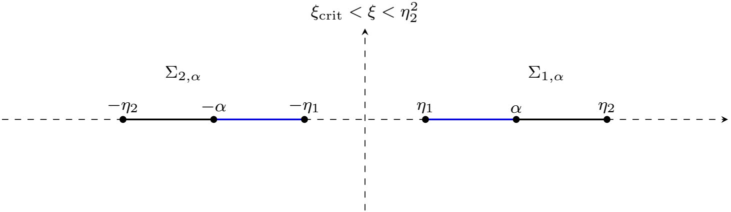

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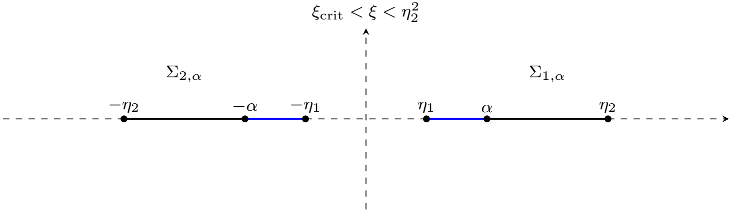

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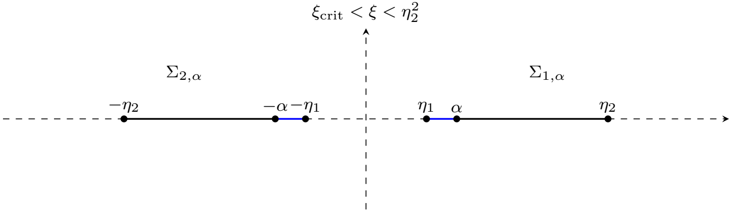

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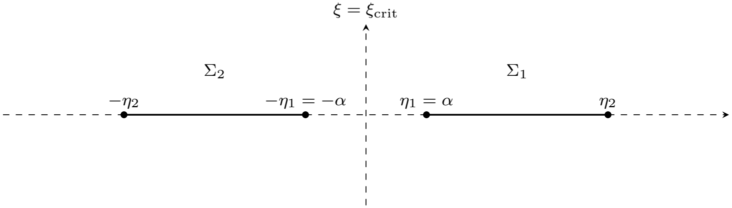

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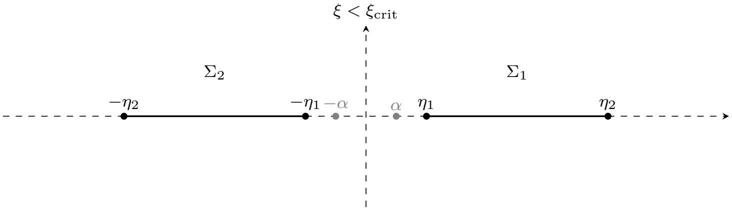

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For $\xi < \xi_{\text{crit}}$ the value of α remains always smaller than η_1.

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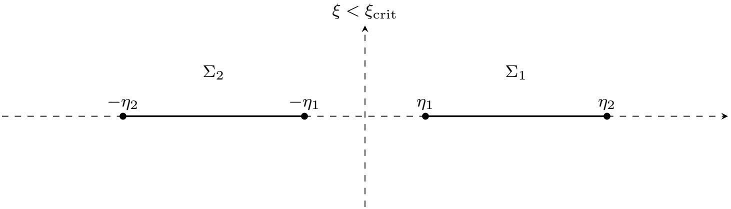

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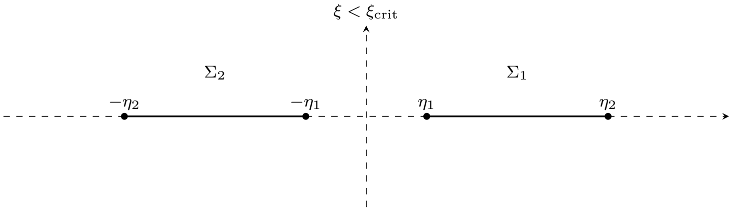


Recipe:

- Similar construction of the model problem, without the α dependency.
- The local parametrices at the endpoints are four Bessel parametrices.

Theorem (G., Grava, McLaughlin, '18)

In the regime $t \nearrow +\infty$, $\xi \leq \xi_{\text{crit}}$, $\frac{t\widetilde{\Omega}+\widetilde{\Delta}}{2\pi i} \neq \frac{2n+1}{2}$, $n \in \mathbb{Z}$, *the potential* $u(x,t)$ *has the following asymptotic expansion*

$$u(x,t) = \eta_2^2 - \eta_1^2 - 2\eta_2^2 \,\mathrm{dn}^2 \left(\eta_2(x - 2(\eta_1^2+\eta_2^2)t + \phi) + K(m) \,|\, m\right) + \mathcal{O}\left(t^{-1}\right) \ ,$$

where $m = \eta_1/\eta_2$, *and*

$$\phi = \int_{\eta_1}^{\eta_2} \frac{\log r(\zeta)}{R_+(\zeta)} \frac{\mathrm{d}\zeta}{\pi i} \ .$$

Conclusion: complete description of a (free) soliton gas potential in the large time regime over the whole real line $x \in \mathbb{R}$.

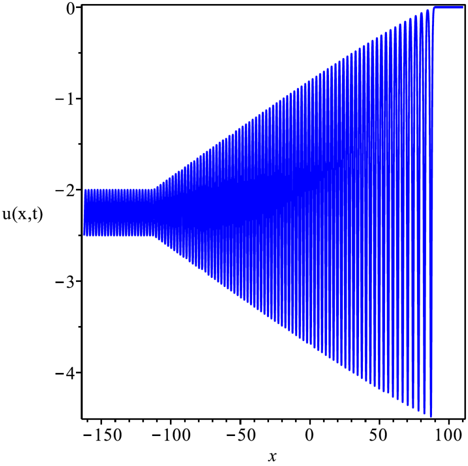


Figure: The asymptotic behaviour of the soliton gas solution. Here $t = 10$, $\eta_1 = 0.5$ and $\eta_2 = 1.5$ and $r(\lambda) \equiv 1$.

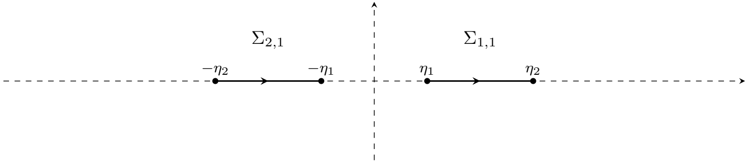
Work in progress and future developments

Reflection coefficients:

1. RH problem with two reflection coefficients r_1 and r_2 (Dyachenko, Zakharov, Zakharov, '16):

$$Y_+(\lambda) = Y_-(\lambda) \begin{cases} \dfrac{1}{1+r_1r_2} \begin{bmatrix} 1-r_1r_2 & -i\,r_2\,e^{2\lambda x} \\ -i\,r_1\,e^{-2\lambda x} & 1-r_1r_2 \end{bmatrix} & \text{on } \Sigma_1 \\ \\ \dfrac{1}{1+r_1r_2} \begin{bmatrix} 1-r_1r_2 & i\,r_1\,e^{2\lambda x} \\ i\,r_2\,e^{-2\lambda x} & 1-r_1r_2 \end{bmatrix} & \text{on } \Sigma_2 \end{cases}$$

2. Multi-band reflection coefficients: the spectral parameter accumulates in two or more disconnected components $\{\Sigma_{1,j} \cup \Sigma_{2,j}\}_{j=1,\dots,M}$



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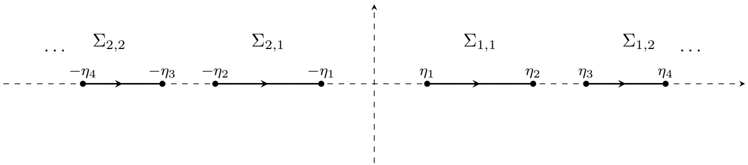

Double scaling limit around the critical values of ξ:

1. What happens in a microscopic neighbourhood of η_2^2?

trivial solution Y	$\longrightarrow$	introduction of a g-function
vanishing of the potential $u(x,t)$	$\longrightarrow$	boundedness of the potential $u(x,t)$

2. What happens in a microscopic neighbourhood of ξ_{crit}?

α-dependent sub-intervals	$\longrightarrow$	full intervals $\Sigma_1 \cup \Sigma_2$
Airy local parametrix	$\longrightarrow$	Bessel local parametrix

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Interaction dynamic:

1. Interaction with another soliton? Numerical experiments by G. El *et al.*

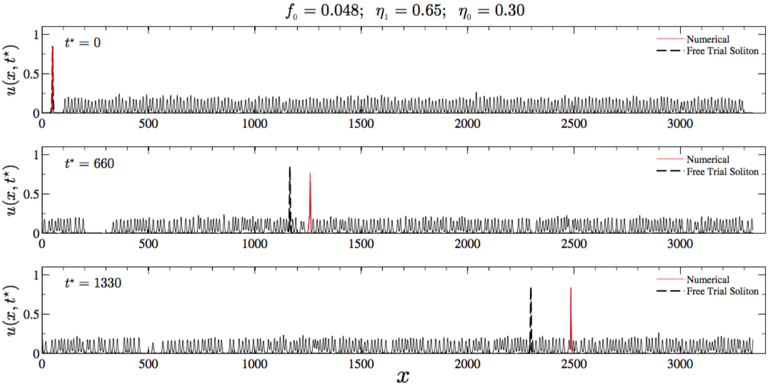

Figure: From *Carbone, Dutyk, El, '16.*

2. Collision with another soliton gas? Numerical experimens by G. El *et al.*

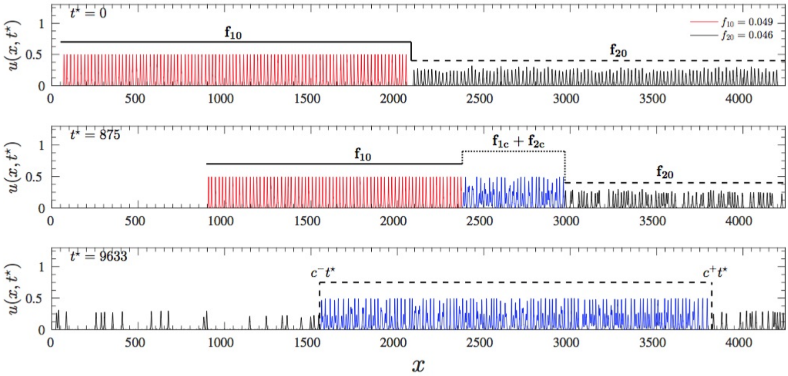

Figure: From *Carbone, Dutyk, El, '16.*

Thank you for your attention!