Manifolds with polynomially convex hull without analytic structure

Alexander J. Izzo

$X \subset \mathbb{C}^n$ compact

Definition: The **polynomially convex hull** of $X \subset \mathbb{C}^n$ is

$$\widehat{X} = \{z \in \mathbb{C}^n : |p(z)| \leq \sup_{x \in X} |p(x)| \quad \text{for every polynomial } p\}.$$

X is said to be **polynomially convex** if $\widehat{X} = X$.

$P(X)$= the uniform closure of the polynomials in $z_1, \ldots, z_n$ on X

The maximal ideal space of $P(X)$ is $\widehat{X}$.

In particular, a necessary condition for $P(X) = C(X)$ is that X be polynomially convex.

Replacing modulus of polynomials with linear functions gives ordinary convexity.

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$\widehat{\partial D} = \overline{D}$ (D the unit disc in $\mathbb{C}$)

In general, for $X \subset \mathbb{C}$, $\widehat{X}$ is obtained from X by filling in the holes, and the functions in $P(X)$ extend to holomorphic functions in the holes.

Examples in $\mathbb{C}^2$:

$X_1 = \{(e^{i\theta}, 0) : 0 \leq \theta \leq 2\pi\}$ $\qquad$ $\widehat{X}_1 = \overline{D} \times \{0\}$

$X_2 = \{(e^{i\theta}, e^{-i\theta}) : 0 \leq \theta \leq 2\pi\}$ $\qquad$ $\widehat{X}_2 = X_2$

Existence of analytic structure in hulls

It was once conjectured that for $X \subset \mathbb{C}^n$, if $\widehat{X} \setminus X$ is nonempty, then $\widehat{X} \setminus X$ contains an analytic disc.

Definition: A set $E \subset \mathbb{C}^n$ **contains an analytic disc** if there is a nonconstant analytic function $\varphi : D \to \mathbb{C}^n$ with $\varphi(D) \subset E$.

Support for conjecture that $\widehat{X} \setminus X \neq \emptyset$ implies $\widehat{X} \setminus X$ contains an analytic disc

Theorem (Wermer, 1958): Suppose X is an analytic curve in $\mathbb{C}^n$. Then $\widehat{X} \setminus X$ is either empty or is a one-dimensional analytic variety.

This theorem was strengthened by Bishop, Royden, Stolzenberg, Alexander, etc.

Theorem (Alexander, 1971): Same result holds for X a rectifiable curve.

Stolzenberg shattered the hope that analytic structure always exists.

Theorem (Stolzenberg, 1963): There exists a compact set X in $\mathbb{C}^2$ such that $\widehat{X} \setminus X$ is nonempty but contains *no* analytic disc.

Henceforth will use the phrase "X has hull without analytic structure" to mean that $\widehat{X} \setminus X$ is nonempty but contains no analytic disc.

Theorem (Basener, 1973): There exists a smooth 3-sphere in $\mathbb{C}^5$ having hull without analytic structure.

$$D = \{z \in \mathbb{C} : |z| < 1\} \qquad \partial D = \{z \in \mathbb{C} : |z| = 1\}$$

Theorem (Wermer, 1982): There exists a compact set contained in $\partial D \times \mathbb{C}$ having hull without analytic structure.

$$B_n = \{z \in \mathbb{C}^n : \|z\| < 1\}$$

Theorem (Duval-Levenberg, 1995): Let K be a compact, polynomially convex subset of B_n, $n \geq 2$. Then there is a compact subset X of ∂B_n such that $\widehat{X} \supset K$ and such that $\widehat{X} \setminus (X \cup K)$ contains no analytic disc.

Theorem (Alexander, 1998): There exists a compact set contained in $\partial D \times \partial D$ having hull without analytic structure.

New Results

Theorem (I., Samuelsson Kalm, Wold; I., Stout): Every smooth compact manifold of real dimension $m \geq 2$ smoothly embeds in $\mathbb{C}^N$ for some N so as to have hull without analytic structure.

When $m \geq 3$, can take $N = 2m + 4$. (I., S. Kalm, Wold)

When $m = 2$, can take $N = 3$. (I., Stout)

Theorem (I.-Stout): Every compact 2-manifold smoothly embeds in $\mathbb{C}^3$ so as to have hull without analytic structure. Furthermore, the embedded manifold can be chosen to be totally real.

Compare

Theorem (Duchamp, Stout 1981): No compact m-dimensional manifold is polynomially convex in $\mathbb{C}^m$.

Theorem (Alexander 1996): Every totally real compact m-dimensional smooth manifold in $\mathbb{C}^m$ has an analytic disc in its hull.

$[f_1, \ldots, f_n]$ =uniformly closed algebra generated by $f_1, \ldots, f_n$

Wermer's maximality theorem (1953): The disc algebra on the circle $P(\partial D)$ is a maximal (closed) subalgebra of $C(\partial D)$, i.e., if $f \in C(\partial D) \setminus P(\partial D)$, then $[z, f] = C(\partial D)$.

Can reformulate as a statement about the graph Γ_f of f: For $f \in C(\partial D)$, either $\widehat{\Gamma}_f \setminus \Gamma_f = \emptyset$ and $P(\Gamma_f) = C(\Gamma_f)$, or else, $\widehat{\Gamma}_f \setminus \Gamma_f$ is an analytic disc.

Viewed in this way, Samuelsson Kalm and Wold began proving analogues in several variables.

$T^2 = \{(z_1, z_2) : |z_1| = |z_2| = 1\}$

Samuelsson Kalm and Wold needed an additional hypothesis in their several variable analogues of Wermer's theorem.

Definition: A complex-valued function on an open set in $\mathbb{C}^n$ is **pluriharmonic** if it is harmonic on each complex line.

Theorem (Samuelsson-Wold 2012): Suppose $f_1, \ldots, f_N \in C(T^2)$ have pluriharmonic extensions to D^2. Then either
(i) $\widehat{\Gamma}_f \setminus \Gamma_f = \emptyset$ and $[z_1, z_2, f_1, \ldots, f_N]_{T^2} = C(T^2)$, or else
(ii) $\widehat{\Gamma}_f \setminus \Gamma_f$ contains as analytic disc.

Can the pluriharmonic hypothesis be dropped? No.

Theorem (I., Samuelsson Kalm, Wold): There exists a real-valued smooth function f on $T^2 =\subset \mathbb{C}^2$ such that the graph $\Gamma_f \subset \mathbb{C}^3$ has a hull without analytic structure.

Proof sketch:

Theorem (Alexander, 1998): There exists a compact set E contained in T^2 having hull without analytic structure.

Lemma: Let $f \in C(X)$ be real-valued, $X \subset \mathbb{C}^n$ compact. Then graph Γ_f of f satisfies $\widehat{\Gamma}_f = \bigcup(\widehat{f^{-1}(t)} \times \{t\}) \subset \mathbb{C}^{n+1}$.

It suffices to construct a real-valued $f \in C^\infty(T^2)$ with zero set E and all other level sets polynomially convex.

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Note: Every closed subset of a smooth manifold is the zero set of some smooth function.

Thus can choose f with zero set E. Arranging for the other level sets to be polynomially convex requires some work. A key ingredient is the following lemma.

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Proof sketch:

It suffices to construct a real-valued $f \in C^\infty(T^2)$ with zero set E and all other level set polynomially convex.

Let $C_a = \{(z_1, z_2) \in T^2 : z_1 = a\}$.

Lemma: Let $K \subset T^2$ be a closed set that contains no full C_a and is disjoint from some C_a. Then $P(K) = C(K)$, and in particular K is polynomially convex.

Theorem (I., Samuelsson Kalm, Wold): Every smooth compact manifold of real dimension $m \geq 3$ smoothly embeds in $\mathbb{C}^{2m+4}$ so as to have hull without analytic structure.

The proof uses Alexander's set in T^2 with hull without analytic structure to get an embedding in some $\mathbb{C}^N$, and a transversality argument to reduce the dimension to $2m + 4$.

What about 2-manifolds with hull without analytic structure?

Theorem (I.-Stout): Every compact 2-manifold smoothly embeds in $\mathbb{C}^3$ so as to have hull without analytic structure. Furthermore, the embedded manifold can be chosen to be totally real.

Classification of compact surfaces: Denote the sphere by $\mathbb{S}$, the torus by $\mathbb{T}$, and the projective plane by $\mathbb{P}$. Denote the connected sum of two compact surfaces S_1 and S_2 by $S_1 \# S_2$. Then the following is a complete list of the compact surfaces:

$$\mathbb{S}; \quad \mathbb{T},\ \mathbb{T}\#\mathbb{T},\ \mathbb{T}\#\mathbb{T}\#\mathbb{T}, \ldots; \quad \mathbb{P},\ \mathbb{P}\#\mathbb{P},\ \mathbb{P}\#\mathbb{P}\#\mathbb{P}, \ldots.$$

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Now to get an embedding of a connected sum of tori in $\mathbb{C}^3$ with hull without analytic structure:
Start with the standard torus $T^2 \subset \mathbb{C}^2$, line up as many disjoint copies of the torus as needed in $\mathbb{C}^2$, cut out small holes, and connect with tubes to form Σ. Then define a smooth real-valued function f on Σ with zero set Alexander's set E and all other level sets polynomially convex. Then invoke the lemma used earlier about the hull of the graph of a real-valued function.

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For the general case, we find a smooth sphere in $\mathbb{C}^2$ containing Alexander's set, and then form an arbitrary surface again by forming a connected sum using tubes.