A new approach to the L^p-theory of $-\Delta + b \cdot \nabla$, and its applications to Feller processes with general drifts.

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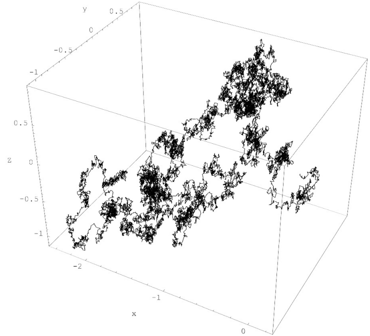
October 2015

The key component of many models of Math. Physics: Brownian motion



A trajectory of a three-dimensional Brownian motion

Brownian motion is modeled by Wiener process W_t (where $t \geqslant 0$, $W_0 = 0$)

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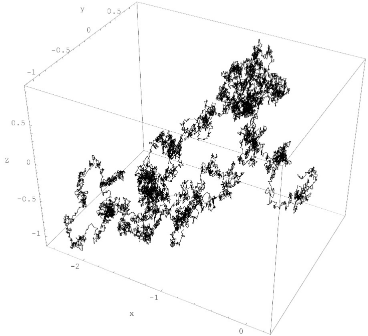

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The simplest diffusion processes

Brownian motion perturbed by a (singular) drift $b : \mathbb{R}^d \to \mathbb{R}^d$?

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The long search for **the cricial singularities** of the drift b ...

A bridge between Probability and Analysis

Probability		**Analysis**
W_t	$\longleftrightarrow$	$-\Delta$

A bridge between Probability and Analysis

Precisely, given a realization $W_s \in \mathbb{R}^d$, $s < t$, we have

$$\underbrace{\mathbb{P}[W_t \in A]}_{\text{Probability}} = \underbrace{e^{(t-s)\Delta}\mathbf{1}_A(W_s)}_{\text{Analysis}}$$

i.e. to find the probability we need to solve the heat equation

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$\Rightarrow$ Analytic methods in Probability

By analogy: let X_t be a Brownian motion perturbed by drift $b : \mathbb{R}^d \to \mathbb{R}^d$
Then we must have

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We can solve the heat equation for fairly singular b's. But ...

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The research program started in 1980s, closely tied to the progress in PDEs, and continuing within emerging areas of Probability (SPDEs)...

What singularities of the drift are admissible?

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$$d \geqslant 3$$

The best possible result in terms of L^p-spaces

Denote $L^p = L^p(\mathbb{R}^d)$

Stampacchia ... Krylov, Röckner, Stannat and many others

$$L^d + L^\infty$$

$$\uparrow$$

$$L^p + L^\infty \quad (p > d)$$

Critical drifts?

Example: A vector field having critical singularity

$$b(x) := x|x|^{-2}, \quad x \in \mathbb{R}^3$$

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Replace $x|x|^{-2}$ with $(1+\varepsilon)x|x|^{-2}$ and the process will cease to exist

In other words, singularities of b are **critical if they are sensitive to multiplication by constants**

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The singularities of a $b \in L^d$ are sub-critical

The classes of critical vector fields previously studied in the literature

The class of form-bounded vector fields[1]

$$\mathbf{F}_\delta := \left\{ b \in L^2_{\mathrm{loc}} : \lim_{\lambda \uparrow \infty} \left\| |b| (\lambda - \Delta)^{-\frac{1}{2}} \right\|_{L^2 \to L^2} \leqslant \sqrt{\delta} \right\}$$

Example: $b(x) = \sqrt{\delta} \frac{d-2}{2} x |x|^{-2}$ (Hardy inequality)

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$\mathbf{F}_\delta$ is 'responsible' for dissipativity of $-\Delta + b \cdot \nabla$ in L^p

$\Rightarrow$ a diffusion via a Moser-type iterative procedure of Kovalenko-Semenov

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The Kato class of vector fields

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Example:

$$|b(x)| = \big| |x| - 1 \big|^{-\gamma}, \quad \gamma < 1$$

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$\mathbf{K}_\delta^{d+1}$ is 'responsible' for the Gaussian bounds[2] for $-\Delta + b \cdot \nabla$

$\Rightarrow$ the Gaussian bounds yield a diffusion

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Classes $\mathbf{K}_\delta^{d+1}$ and $\mathbf{F}_\delta$ play prominent role in Analysis

Crucial features:

– Defined in terms of the operators that constitute the problem[3]

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– What really matters is the relative bound $\delta > 0$ (as in $\delta x|x|^{-2}$; has to be small, so that $b \cdot \nabla \leq -\Delta$). For instance, $L^d \subset \mathbf{F}_0 := \bigcap_{\delta>0} \mathbf{F}_\delta$.

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– Are L^1, L^2-conditions (e.g. Kato class of measure-valued drifts: Bass-Chen, Kim-Song), cf. "$b \in L^d$"

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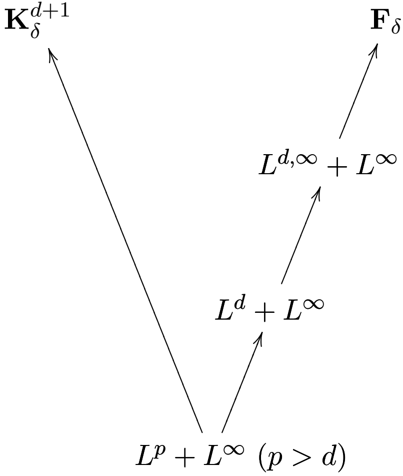
The state of affairs (not so long ago)



The search for 'the right' class of critical drifts b

The next step:

$$b := b_1 + b_2, \qquad b_1 \in \mathbf{K}_\delta^{d+1},\ b_2 \in \mathbf{F}_\delta$$

(i.e. b combines critical point and critical hypersurface singularities)

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The main obstacle: "$b \in \mathbf{F}_\delta$" destroys Gaussian bounds, and "$b \in \mathbf{K}_\delta^{d+1}$" destroys L^p-dissipativity (crucial for the existing proofs)

To summarize (so far)

1. The two prominent classes of singular vector fields $\mathbf{K}_\delta^{d+1}$, $\mathbf{F}_\delta$ are responsible for two fundamental properties of $-\Delta + b \cdot \nabla$:

“Gaussian bounds”, “dissipativity”

(both imply that $-\Delta + b \cdot \nabla$ generates a diffusion)

To summarize (so far)

1. The two prominent classes of singular vector fields $\mathbf{K}_\delta^{d+1}$, $\mathbf{F}_\delta$ are responsible for two fundamental properties of $-\Delta + b \cdot \nabla$:

"Gaussian bounds", "dissipativity"

(both imply that $-\Delta + b \cdot \nabla$ generates a diffusion)

2. It is clear that neither $\mathbf{K}_\delta^{d+1}$ nor $\mathbf{F}_\delta$ is responsible for the property

"to generate a diffusion"

Part II: "A new hope"

arXiv:1502.07286, arxiv:1508:05983[4]

[4] Or www.math.toronto.edu/dkinz

From an analyst perspective

$-\Delta + b \cdot \nabla$ generates a diffusion if it generates a strongly continuous semigroup in the Banach space $C_\infty := \{f \in C(\mathbb{R}^d) : f \text{ vanishes at } \infty\}$

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$-\Delta + b \cdot \nabla$ generates a diffusion if it generates a strongly continuous semigroup in the Banach space $C_\infty := \{f \in C(\mathbb{R}^d) : f \text{ vanishes at } \infty\}$

In other words, we solve the Cauchy problem

$$\partial_t u - \Delta u + b \cdot \nabla u = 0, \qquad u(0,\cdot) = f(\cdot) \in C_\infty$$

in C_∞, i.e. we must have strong continuity:

$$\lim_{t \downarrow 0} u(t,\cdot) = f(\cdot) \quad \text{in } C_\infty$$

A theorem in Probability

Strong continuity property in $C_\infty \Rightarrow$ the fundamental solution of $-\Delta + b \cdot \nabla$ is the transition (sub-) probability function of a diffusion

... a bridge between Probability and Analysis

The class of weakly form-bounded vector fields

$$\mathbf{F}_\delta^{\frac{1}{2}} := \{b \in L^1_{\rm loc} : \big\||b|^{\frac{1}{2}}(\lambda-\Delta)^{-\frac{1}{4}}\big\|_{2\to 2} \leqslant \delta\},$$

Weakly form-bounded vector fields

Proposition:

$$\mathbf{F}_{\delta_1} + \mathbf{K}^{d+1}_{\delta_2} \subsetneq \mathbf{F}^{\frac{1}{2}}_{\delta}, \quad \delta := \delta_1 + \delta_2$$

Proof (easy): interpolation, Heinz-Kato inequality

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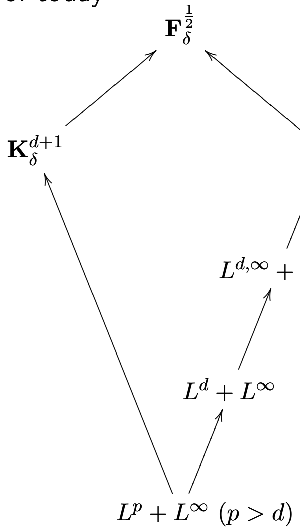
Corollary:

$\mathbf{F}^{\frac{1}{2}}_{\delta}$ allows to combine critical point and critical hypersurface singularities

Weakly form-bounded vector fields

The state of affairs as of today



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L^2-regularity theory of $-\Delta + b \cdot \nabla$, $b \in \mathbf{F}_\delta^{\frac{1}{2}}$ (JFA, Semenov, 2006)

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Even in L^2: KLMN theorem doesnt't apply

Let $b \in \mathbf{F}_\delta^{\frac{1}{2}}$

We need: an operator realization $\Lambda(b)$ of $-\Delta + b \cdot \nabla$ generating a (positivity preserving, contraction) C_0-semigroup $T_t \in \mathcal{B}(C_\infty)$, i.e.

(1) $T_{t+s} = T_t T_s$, $T_0 = 1$

(2) $T_t f \overset{s}{\rightarrow} T_s$ in C_∞ as $t \rightarrow s$, $s \geqslant 0$.

(3) $\frac{d}{dt} T_t f = \Lambda(b) T_t f$

C_0-semigroups (Kato, Yosida ...)

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Precise meaning of 'generating':

$$\Lambda(b) f := \lim_{t \downarrow 0} \frac{T_t f - f}{t}, \quad f \in C_\infty$$

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$$\Lambda(b) f := \lim_{t \downarrow 0} \frac{T_t f - f}{t}, \quad f \in C_\infty$$

Denote $e^{-t\Lambda(b)} := T_t$

We will 'design' the resolvent

$$(\lambda + \Lambda(b))^{-1} \in \mathcal{B}(C_\infty), \quad \lambda > \lambda_0 > 0$$

of the required diffusion generator $\Lambda(b)$

A new approach: ‘designing the resolvent’

Our starting object: an operator-valued function on L^p, p is in a bounded open interval depending on the relative bound δ,

$$\Theta_p(\lambda, b) := (\lambda - \Delta)^{-1} - (\lambda - \Delta)^{-\frac{1}{2}} Q_p (1 + T_p)^{-1} G_p,$$

where

$$Q_p = (\lambda - \Delta)^{-\frac{1}{2}} |b|^{\frac{1}{p'}},$$

$$T_p = b^{\frac{1}{p}} \cdot \nabla (\lambda - \Delta)^{-1} |b|^{\frac{1}{p'}},$$

$$G_p = b^{\frac{1}{p}} \cdot \nabla (\lambda - \Delta)^{-1}, \qquad b^{\frac{1}{p}} := b|b|^{\frac{1}{p}-1}$$

Formally,

$$\Theta_p(\lambda, b) = \sum_{k=0}^{\infty} (-1)^k (\lambda - \Delta)^{-1} \underbrace{b \cdot \nabla (\lambda - \Delta)^{-1} \dots b \cdot \nabla (\lambda - \Delta)^{-1}}_{k \text{ times}}$$

where the RHS is the Neumann series for $(\lambda + \Lambda(b))^{-1}$

So, $\Theta_p(\lambda, b)$ is 'a candidate' for the resolvent $(\lambda + \Lambda(b))^{-1}$!

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Proposition: If $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, then Q_p, T_p, $G_p \in \mathcal{B}(L^p)$

Proof: Using L^p-inequalitites between $(\lambda - \Delta)^{\frac{1}{2}}$ and 'potential' $|b|$ (Liskevich-Semenov, 1996)[5]

[5]Note: $\mathbf{K}_\delta^{d+1}$, $\mathbf{F}_\delta$ reduce everything to $-\Delta + b^2$

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The key insight:

If the relative bound $\delta > 0$ (in $b \in \mathbf{F}_{\delta}^{\frac{1}{2}}$) is small, we can select $p > d$

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So,

$$\Theta_p(\lambda, b) L^p \subset C_\infty$$

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Now, to prove that

$$\Theta_p(\lambda, b_n)f \xrightarrow{s} \Theta_p(\lambda, b)f \quad \text{in } C_\infty, \qquad f \in C_0^\infty,$$

where b_n are bounded (smooth) approximations of b, we only need to work in L^p, $p > d$, **a space having much weaker topology**

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$\Rightarrow$ the gain in the admissible singularities of the drift

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[6] Bass-Chen [Ann. Prob. 2003], Chen-Kin-Song [Ann. Prob. 2012]

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If we stay[6] in $C_\infty \Rightarrow b \in \mathbf{K}_\delta^{d+1}$

Note: $\mathbf{K}_0^{d+1}$ ensures continuity of $\nabla e^{t\Lambda(b)}$

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This method:

1. Also provides a detailed L^p-regularity of $-\Delta + b \cdot \nabla$, e.g. characterizes smoothness of the domain of the generator in terms of $\delta > 0$ (in "$b \in \mathbf{F}_\delta^{\frac{1}{2}}$")

Concluding remarks

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4. ...

$$\|f\|_{p,\infty} := \left(\sup_{t>0}\ t^p \mu\{|f(x)| > t\}\right)^{\frac{1}{p}}$$

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Since $\Theta_p(\lambda, b_n)f \xrightarrow{s} \Theta_p(\lambda, b)f$ in C_∞, $\quad f \in C_0^\infty$,

$$\underbrace{\|\Theta_p(\lambda, b_n)\|_{L^\infty \to L^\infty} \leqslant \lambda^{-1}}_{\text{'external' fact}} \quad \Rightarrow \quad \|\Theta_p(\lambda, b)f\|_{L^\infty} \leqslant \lambda^{-1}\|f\|_{L_\infty},$$

so we have a well defined the 'true candidate' for the resolvent:

$$\Theta(\lambda, b) := \big(\Theta_p(\lambda, b)|_{L^p \cap C_\infty}\big)^{\mathrm{cl}}_{C_\infty} \in \mathcal{B}(C_\infty)$$

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$\Theta(\lambda, b)$ satisfies (same argument with the Sobolev embedding theorem)

$$\lambda\Theta(\lambda, b) \xrightarrow{s} 1 \quad \text{in } C_\infty \text{ as } \lambda \uparrow \infty$$

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$$\lambda\Theta(\lambda, b) \xrightarrow{s} 1 \quad \text{in } C_\infty \text{ as } \lambda \uparrow \infty$$

$\Rightarrow$ a pseudoresolvent $\Theta(\lambda, b)$ is the resolvent of a densely defined operator

Now,

$$\|\lambda\Theta(\lambda, b)\|_{L^\infty \to L^\infty} \leqslant 1$$

(proved in the last slide) $\Rightarrow$ we can **define** $(\lambda + \Lambda(b))^{-1} := \Theta(\lambda, b)$

$\square$