Cyclic polynomials in 2 variables

(joint with Beneteau, Kosinski, Liaw, Seco, Sola)

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FUNDAMENTAL QUESTION IN OPERATOR THEORY:

GIVEN $T \in \mathcal{B}(\mathcal{H})$ (OR $T = (T_1, T_2, \ldots, T_n)$)

AND $[v] = \overline{\text{span}}\{p(T)v : p = \text{polynomial}\}$

WHEN IS v <u>CYCLIC</u>?

$\Leftrightarrow$ WHEN IS $[v] = \mathcal{H}$?

CLASSICAL EXAMPLE

LET $H^2(\mathbb{D})$ = HARDY SPACE ON UNIT DISK

$$= \Big\{ \sum_{n\geq 0} a_n z^n : \sum |a_n|^2 < \infty \Big\}$$

$$\cong \{ f \in L^2(\mathbb{T}) : \hat{f}(n) = 0 \ \forall n < 0 \}$$

$M_z : H^2 \to H^2 \quad M_z f = zf$

WHEN IS $f \in H^2$ CYCLIC FOR M_z?

- NEC. & SUFF. TO SHOW

  $\exists\, p_n$ polynomials s.t. $p_n f \longrightarrow 1$ IN H^2

- f CYCLIC $\Longrightarrow f(z) \neq 0 \ \forall z \in \mathbb{D}$

  $\big(|p_n(z) f(z) - 1| \leq C_z \| p_n f - 1 \|_{H^2} \big)$

$f(z) \neq 0 \;\; \forall z \in \mathbb{D}$ NOT SUFFICIENT

BEURLING'S THEOREM:

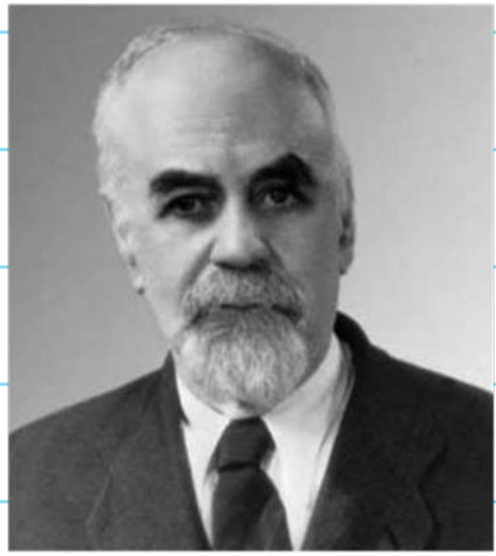
$f \in H^2$ IS CYCLIC FOR M_z

IFF

f IS OUTER:

$$f(0) \neq 0 \;\; \& \;\; \log|f(0)| = \int_{\mathbb{T}} \log|f| \, d\sigma$$

$$\left(\Leftrightarrow f(z) = \exp \int_{\mathbb{T}} \log|f| \frac{z+\zeta}{z-\zeta} \, |d\zeta| \right)$$



THIS IS REALLY MY RESULT FROM 1932

EX: $\mathcal{D}$ = DIRICHLET SPACE $= \left\{ \sum_{n\geq 0} a_n z^n : \sum (n+1)|a_n|^2 < \infty \right\}$

WHEN IS f CYCLIC FOR $M_z : \mathcal{D} \to \mathcal{D}$?

BROWN-SHIELDS CONJECTURE: $\iff$ f IS OUTER & $\{ z \in \mathbb{T} : f(z) = 0 \}$ HAS LOG-CAPACITY ZERO.

EX: $H^2(\mathbb{D}^n) = \left\{ \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha z^\alpha : \sum |a_\alpha|^2 < \infty \right\}$

$T = (M_{z_1}, M_{z_2}, \dots, M_{z_n})$

WHEN IS $f \in H^2(\mathbb{D}^n)$ CYCLIC FOR T ?

OPEN QUESTION

GOAL: CHARACTERIZE CYCLIC POLYNOMIALS FOR M_{z_1}, M_{z_2} ON A RANGE OF SPACES....

$\mathcal{D}_\alpha$ = DIRICHLET TYPE SPACES

$$= \left\{ \sum_{j,k \geq 0} a_{j,k} z_1^j z_2^k : \sum (j+1)^\alpha (k+1)^\alpha |a_{j,k}|^2 < \infty \right\}$$

$$\subset \mathrm{Hol}(\mathbb{D}^2)$$

$\alpha = -1 \longrightarrow$ BERGMAN

$\alpha = 0 \longrightarrow$ HARDY

$\alpha = 1 \longrightarrow$ DIRICHLET

$\mathcal{D}_\alpha \supset \mathcal{D}_\beta \quad \alpha < \beta$

RECALL: f BEING CYCLIC IN $\mathcal{D}_\alpha$

MEANS IT IS "APPROXIMATELY INVERTIBLE"

IN THE SENSE THAT

$\exists$ p_n polynomials SUCH THAT

$$p_n f \to 1 \text{ IN } \mathcal{D}_\alpha$$

LARGER α MEANS MORE REGULARITY ON $\mathbb{T}^2$ AND BOUNDARY VALUES PLAY A BIGGER ROLE

- $f \in \mathbb{C}[z_1, z_2]$ CYCLIC FOR (M_{z_1}, M_{z_2}) ON $\mathcal{D}_\alpha \Rightarrow f(z_1,z_2) \neq 0$ $(z_1, z_2) \in \mathbb{D}^2$.

- $f(z) \neq 0$ ON $\mathbb{D}^2 \Rightarrow f$ CYCLIC IN $H^2(\mathbb{D}^2)$ (GINSBURG, NEUWIRTH, NEWMAN 1970)

(Key idea: $\left|\frac{f(z)}{f(rz)}\right| \leq 2^{\sum \deg_{z_i} f}$)

MAIN THEOREM: Let $f \in \mathbb{C}[z_1, z_2]$ AND ASSUME: $f(z) \neq 0 \ \forall z \in \mathbb{D}^2$, f IRREDUCIBLE

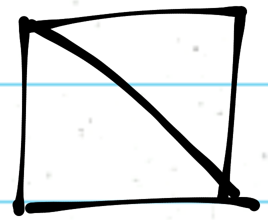
THEN:

- f IS CYCLIC FOR M_{z_1}, M_{z_2} IN $\mathcal{D}_\alpha$ FOR $\alpha \leq 1/2$
- IF f HAS FINITELY MANY ZEROS ON $\mathbb{T}^2$ OR DEPENDS ON 1 VAR, THEN f IS CYCLIC FOR $\alpha \leq 1$.
- IF f HAS INFINITELY MANY ZEROS ON $\mathbb{T}^2$ THEN f IS NON-CYCLIC FOR $\alpha > 1/2$.

(• f CYCLIC FOR $\alpha > 1 \iff$ NO ZEROS IN $\overline{\mathbb{D}^2}$)

BASIC EXAMPLES

(1) $f(z_1, z_2) = 1 - z_1 z_2$



ZERO SET ON $\mathbb{T}^2$

CYCLIC IN $\mathcal{D}_\alpha \iff \alpha \le 1/2$

(2) $g(z_1, z_2) = 2 - z_1 - z_2$

CYCLIC IN $\mathcal{D}_\alpha \iff \alpha \le 1$

(3) $h(z_1, z_2) = 1 - z_1$

CYCLIC IN $\mathcal{D}_\alpha \iff \alpha \le 1$

EASY PARTS

- f DEPENDING ON 1 VAR .... CHECK DIRECTLY

  CYCLIC IN $\mathcal{D}_\alpha$ FOR $\alpha \leq 1$

- f w/ FINITELY MANY ZEROS ON $\overline{\mathbb{D}}^2$

  BY ŁOJASIEWICZ INEQUALITY

  $\exists\, p \in \mathbb{C}[z]$ w/ NO ZEROS IN $\mathbb{D}$ S.T.

  $p/f \in \mathcal{D}_\alpha \implies p \in f\mathcal{D}_\alpha \quad \therefore f\mathcal{D}_\alpha$ DENSE $\alpha \leq 1$.

DIFFICULT PARTS:

- f W/ INFINITELY MANY ZEROS ON $\mathbb{T}^2$

  USES DETERMINANTAL REPRESENTATION

$$f(z_1, z_2) = \underset{\substack{\uparrow \\ \text{CONSTANT}}}{c} \det\left(I - \underset{\substack{\uparrow \\ \text{UNITARY} \\ \text{MATRIX}}}{U} \begin{pmatrix} z_1 I & \\ & z_2 I \end{pmatrix}\right)$$

TO SHOW f CYCLIC FOR $\alpha \leq 1/2$.

- f w/ INFINITELY MANY ZEROS ON $\mathbb{T}^2$ IS NON-CYCLIC FOR $\alpha > 1/2$ PROVEN VIA

FACT:

$$\mathrm{cap}_\alpha(Z_f \cap \mathbb{T}^2) > 0 \text{ FOR } \alpha > 1/2$$
$$\Rightarrow f \text{ NON-CYCLIC}$$

$$\mathrm{cap}_\alpha(E) = \left(\inf\{ I_\alpha[\mu] : \mu \in \mathcal{P}(E) \}\right)^{-1}$$

$$I_\alpha[\mu] = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \frac{1}{|z_1 - w_1|^{1-\alpha}} \frac{1}{|z_2 - w_2|^{1-\alpha}} \, d\mu(z) \, d\mu(w)$$

ILLUSTRATION OF WHERE $\alpha = 1/2$ PLAYS A ROLE:

$f(z_1, z_2) = 1 - z_1 z_2$

SUPPOSE $g \perp (1 - z_1 z_2)\mathbb{C}[z_1, z_2]$ IN $\mathcal{D}_\alpha$

(WANT $g = 0$, TO CONCLUDE f CYCLIC)

$$g \perp (1 - z_1^k z_2^k) \quad \forall k \geq 1$$

$$|\langle g, 1\rangle_{\mathcal{D}_\alpha}|^2 = |\langle g, z_1^k z_2^k\rangle_{\mathcal{D}_\alpha}|^2 \quad \forall k \geq 1$$

$$\downarrow \qquad\qquad \downarrow$$

$$|\hat{g}(0,0)|^2 = |(1+k)^{2\alpha}\ \hat{g}(k,k)|^2$$

$$\sum_{k \geq 1} \frac{|\hat{g}(0,0)|^2}{(1+k)^{2\alpha}} = \sum_{k \geq 1} (1+k)^{2\alpha} |\hat{g}(k,k)|^2$$

$$\leq \|g\|^2_{\mathcal{D}_\alpha} < \infty$$

$$2\alpha \leq 1 \implies \hat{g}(0,0) = 0 \quad \left(\begin{array}{c}\text{OTHER} \\ \text{COEFFICIENTS} \\ \text{SIMILAR}\end{array}\right)$$

CONJECTURE FOR 3 VARS?

$f \in \mathbb{C}[z_1, z_2, z_3]$, $f(z) \neq 0 \ \forall z \in \mathbb{D}^3$

$Z_f \cap \mathbb{T}^3 \neq \emptyset$

f DEPENDING ON ALL 3 VARS

IF $\dim Z_f \cap \mathbb{T}^3 = 2$

CYCLIC IN $\mathfrak{D}_\alpha \iff \alpha \leq 1/3$

IF $\dim Z_f \cap \mathbb{T}^3 = 1$

CYCLIC IN $\mathfrak{D}_\alpha \iff \alpha \leq 1/2$??

✓ IF $\dim Z_f \cap \mathbb{T}^3 = 0$, CYCLIC $\iff \alpha \leq 1$

THANK

YOU!

$$\mathcal{D}_{(\alpha_1,\alpha_2)} = \left\{ f : \sum (j+1)^{\alpha_1} (k+1)^{\alpha_2} |a_{j,k}|^2 < \infty \right\}$$

BEHAVES SIMILARLY

$\alpha_1 + \alpha_2 = 1$ IS THE CRITICAL CUTOFF.