Quadrature Domains and Equilibrium on the Sphere

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MWAA
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- Our interest will be in trying to use the sphere S^2 as the setting for a problem familiar from 2D potential theory:
- How does a charge placed on a conductor distribute to obtain a configuration of minimal energy, in the presence of an electric field?
- In the plane, with logarithmic interactions between charges, if a charge is placed onto a domain $\Omega \subset \mathbb{C}$, what is the equilibrium distribution of the charge?

For example, on a bounded smooth finitely connected domain Ω, we expect the charge to repulse itself as far as possible and reside only on the outer boundary.

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The minimal energy problem is to determine which positive unit Borel measure μ supported in $\bar{\Omega}$ will minimize I_{μ}.

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There are general results about how to identify the equilibrium measure, and how to identify its particular properties.

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This problem can also be analyzed in the plane with logarithmic potential theory, notably as presented by Saff and Totik in their book on the subject.

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The equilibrium measure in the presence of Q is then the positive unit Borel measure which will minimize the weighted energy I_μ^Q.

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- *For some constant F_Q, $U^{\mu_Q}(z) + Q(z) = F_Q$ on supp(μ_Q), and $U_{\mu_Q}(z) + Q(z) \geq F_Q$ on all of $\mathbb{C}$ (q.e.).*

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Note: the equilibrium measure has constant 'weighted potential' on its support. This is intuitively appealing, since if there were a potential difference, a current would flow to equalize it.

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- which is free to redistribute in the presence of an external field
- what is the equilibrium charge configuration given an external field Q?

In this case, we can recover the planar problem via stereographic projection: surface measure on the sphere corresponds to the measure $\frac{dA}{(1+|z|^2)^2}$ in the plane, when the north pole of the sphere is mapped to ∞.

Brauchart, Dragnev, Saff, and Womersley were able to determine what happens when the external field consists of finitely many point charges that are sufficiently separated.

Their result is the following (paraphrasing):

Theorem

(BDSW) Let a unit charge be placed uniformly on the sphere, and then place finitely many sufficiently separated point charges on the sphere (with logarithmic interactions). The energy-minimizing charge distribution in the presence of these point charges is still uniform, but the support excludes perfect spherical caps centered on the point charges. The size of the caps can be explicitly calculated.

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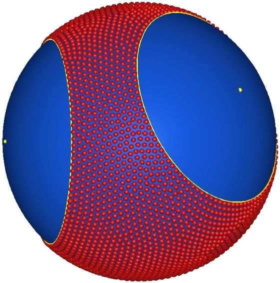
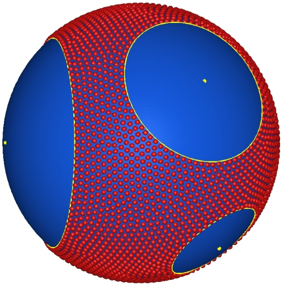
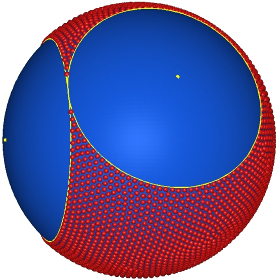
In other words, each point charge has a 'cap of influence' where it tends to repulse the charge on the sphere. 'Sufficiently separated' means that the caps should be disjoint.

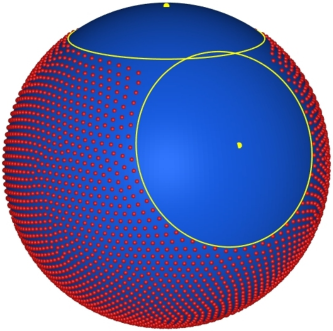
Images from Brauchart, Dragnev, Saff, and Womersely: Logarithmic and Riesz Equilibrium for Multiple Sources on the Sphere: The Exceptional Case, Contemporary Computational Mathematics-A celebration of the 80th Birthday of Ian Sloan, 179–203.







My interest in the problem comes from a question raised in their paper:



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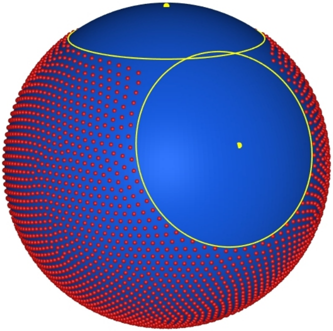

What's going on here?

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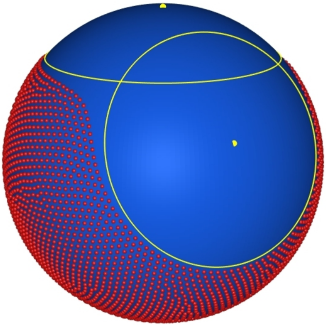

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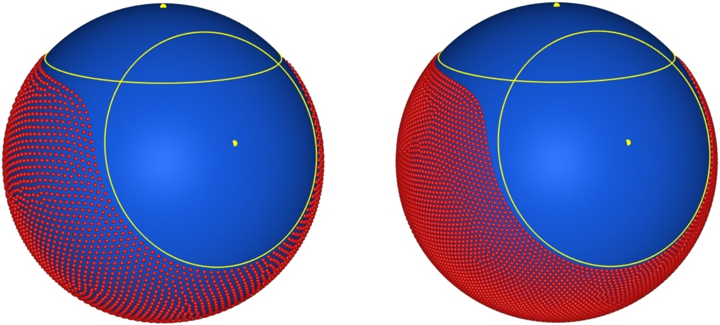
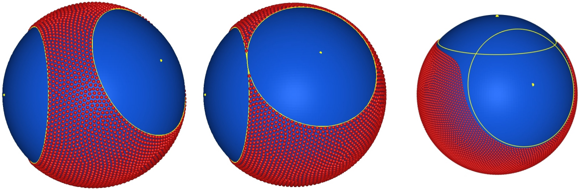


It appears that the moment after caps of influence overlap, the equilibrium support smooths out into a lobed shape. Can we describe the shape?



In the plane, when two circles combine in a potential theory setting and smooth out into a single curve, it makes one think of a Neumann Oval, which is a type of Quadrature Domain.

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There are corresponding shapes for any number of nodes.

A Quadrature Domain is a domain Ω in the plane where integrating functions in a given test class (usually, harmonic L^1, Bergman Space A^2, integrable analytic AL^1, etc.) coincides with taking a linear combination of point evaluations of the functions and their derivatives. The same coefficients and points should work for any function in the test class:

$$\int_\Omega h(z)dA = \sum_{i,j} c_{i,j} h^{(j)}(z_i).$$

The number of terms in the sum is the 'order' of the Q.D., and the points of evaluation z_i are called the 'nodes.'

Examples of Q.D.'s:
Disc (the only Q.D. of order 1)
Neumann Oval (order 2 with distinct nodes)
Cardioid/Limacon (order 2 with single node)
Several overlapped discs merged together in the right way

These domains are very appealing, because they automatically enjoy many strong properties.

- Algebraic Boundary
- Algebraic proper maps to the unit disc
- Maps between Quadrature Domains are algebraic
- Algebraic Bergman Kernel
- Meromorphic Schwarz Function

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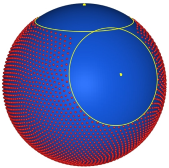
$\int_\Omega hdz \wedge d\bar{z} = \int_{\partial\Omega} \bar{z}h(z)dz = \int_{\partial\Omega} h(z)S(z)dz$. Since S is meromorphic, this just becomes a linear combination of evaluations of h and its derivatives by the Residue Theorem.

Back to the problem at hand:



This region of charge exclusion looks like a Neumann Oval (Q.D.) which was stereographically projected onto S^2. And it was made in a similar way, by overlapping two circles in a potential-theoretic way.

That turns out to be a good inspiration. In fact, under certain assumptions we can prove that the region of exclusion is the projection of a Q.D., using classical Complex Analysis after a stereographic projection.

Assume the complement of the equilibrium support is smooth and simply connected. Rotate the sphere so that the north pole is in the equilibrium support. Then project stereographically to the plane. As described in the BDSW paper, we can write a planar formulation of the Frostman condition of constant weighted potential on the equilibrium measure.

Let $\Omega \subset \mathbb{C}$ be the stereographically projected support of the equilibrium measure (unbounded by our choice of north pole), and assume its complement (the projected region of charge exclusion) is smooth and simply connected. The Frostman condition will involve a potential, the external field from the 2 point charges, and some pieces arising from the distortion caused by the stereographic projection.

Let the point charges be q_1, q_2, and let their planar projections be located at $z_1, z_2 \in \mathbb{C}$, $Q = q_1 + q_2$. Ω will satisfy:

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$$\frac{Q+1}{\pi} \int_\Omega \frac{1}{(1+|w|^2)^2} \ln \frac{1}{|z-w|} dA_w + q_1 \ln \frac{1}{|z-z_1|} + q_2 \ln \frac{1}{|z-z_2|}$$

$$+ (Q+1) \ln \sqrt{1+|z|^2} = const,$$

valid for $z \in \Omega$.

That can be rearranged, by considering the integral as occurring within a large radius that goes to ∞. Differentiate in z, and use Green's Theorem in the form of the C^{∞} version of the Cauchy Integral Formula. After some algebra, and noticing that some terms vanish as the radius goes to ∞, you'll get:

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$\frac{1}{2\pi i}\int_{\partial\Omega^c}\frac{1}{w+\frac{1}{\bar{w}}}\frac{1}{w-z}dw = \frac{q_1}{Q+1}\frac{1}{z_1-z}+\frac{q_2}{Q+1}\frac{1}{z_2-z}$.

This formula itself can be differentiated in z however many times we want: $\frac{1}{2\pi i}\int_{\partial\Omega^c}\frac{1}{w+\frac{1}{\bar{w}}}\frac{1}{(w-z)^m}dw = \frac{q_1}{Q+1}\frac{1}{(z_1-z)^m} + \frac{q_2}{Q+1}\frac{1}{(z_2-z)^m}$.
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This holds for positive integers m.
But now by linearity/partial fractions, notice that this amounts to saying that for any rational function R with poles in Ω:

$$\frac{1}{2\pi i}\int_{\partial\Omega^c}\frac{1}{w+\frac{1}{\bar{w}}}R(w)dw = \frac{q_1}{Q+1}R(z_1) + \frac{q_2}{Q+1}R(z_2).$$

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That's starting to look like a quadrature domain with 2 nodes! To confirm our suspicion, use Mergelyan's Theorem. The rational R in the above equality can be replaced by uniform approximation to be any $h \in A^\infty(\Omega^c)$. Then write the function values on the right side in terms of the Cauchy Integral Formula, rearrange and get:

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$$\int_{\partial\Omega^c}\left(\frac{1}{w+\frac{1}{\bar{w}}}-\frac{1}{w-z_1}-\frac{1}{w-z_2}\right)h(w)dw = 0.$$

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for any $w \in \partial\Omega^c$.
Now solve for $\bar{w}$: it has the boundary values of a meromorphic function. In other words, Ω^c is a Quadrature Domain!

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In other words, our initial suspicion was correct-the region of charge exclusion on the sphere is the stereographic preimage of a quadrature domain of order 2: a Neumann Oval.

With some extra bookkeeping in the calculations, the same idea will show that: if you assume there are finitely many point charges, and the equilibrium support is smooth and connected, then after a stereographic projection, the region of charge exclusion is a union of Quadrature Domains, and the sum of their orders is the number of point charges.

This sheds some extra light on the BDSW case as well-for a single point charge, the equilibrium support excludes a spherical cap because a disc is the only possible Quadrature Domain of order 1.

Thanks!