Error bounds in Fourier extension approximations

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joint work with Jeff Geronimo

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Approximation by truncated Fourier series

Let $f : [-1/2, 1/2] \to \mathbb{C}$. Its truncated Fourier series of length M is

$$f_M(x) = \sum_{(-M-1)/2 \leq k \leq (M-1)/2} a_k e^{2\pi i k x},$$

where

$$a_k = \int_{-1/2}^{1/2} f(x) e^{-2\pi i k x} dx.$$

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This is a great way to approximate functions!

Approximation by truncated Fourier series

In particular, if $f(x)$ is an analytic periodic function, i.e., $f(-1/2) = f(1/2)$, then $f_M(x)$ converges to $f(x)$ uniformly exponentially fast in M:

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- $f(x)$ is not analytic?

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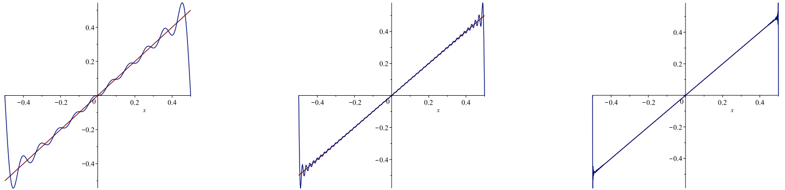
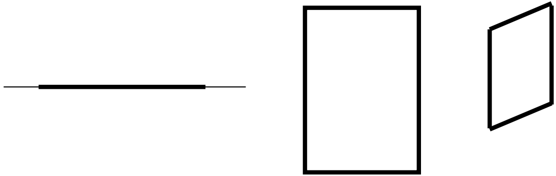
What to do if:

- $f(x)$ is not analytic? No big deal. If $f \in C^\infty$, we still get super-algebraic convergence.
- $f(x)$ is not periodic? This is a problem. We do not get uniform convergence at all (Gibbs phemomenon):



Another issue with Fourier approximations

In more general (higher-dimension) case, if $\Omega \in \mathbb{R}^d$ and $f : \Omega \to \mathbb{C}$, the Fourier series seems to only apply when Ω has rather simple geometry:



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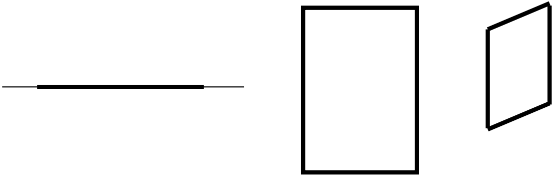
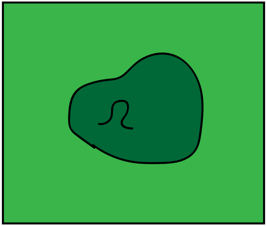


One way to deal with it: extend f smoothly to a periodic function on a larger rectangular domain



Fourier extensions

It is well known that any smooth function f can be extended smoothly to a periodic one $\tilde{f}$ on a larger domain (Whitney, 1934). In fact there are many such extensions.

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The simple idea of the Fourier extension (continuation) method is then to find the truncated Fourier series for $\tilde{f}$. It converges super-algebraically fast to $\tilde{f}$, and $\tilde{f} \equiv f$ on the original (smaller) domain.

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Modern use of this method in numerical analysis:

- Begins with Boyd (2002) and Bruno (2003)
- Applications to PDE's and imaging: Bruno, Han, Pohlman (2007); Bruno and Lyon (2009); Bruno and Albin (2011),
- Stability and convergence considerations: Huybrechs (2010); Adcock and Huybrechs (2014); Adcock, Huybrechs and Martín-Vaquero (2014)

How to use the (1d) Fourier extension method

We have $f : [-1/2, 1/2] \to \mathbb{C}$ smooth but not periodic. Choose $b > 1$, and extend f to $\tilde{f} : [-b/2, b/2] \to \mathbb{C}$ where $\tilde{f}(-b/2) = \tilde{f}(b/2)$. Then use the truncated Fourier series for $\tilde{f}$.

In practice this means we project f onto the space

$$S_M^b = \{e^{\frac{2\pi ik}{b}x}\}_{k\in t(M)}.$$

It is natural to project in $L^2[-1/2, 1/2]$. Define

$$f_M^c := \operatorname{argmin}\{||q - f||_{L^2} : q \in S_M^b\}$$

We call f_M^c the approximation by continuous Fourier extension. It is very accurate.

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$$S_M^b = \{e^{\frac{2\pi i k}{b} x}\}_{k \in t(M)}.$$

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$$f_M^c := \mathrm{argmin}\{||q - f||_{L^2} : q \in S_M^b\}$$

We call f_M^c the approximation by continuous Fourier extension. It is very accurate. Indeed, for all "sufficiently analytic" functions f (Huybrechs, Adcock 2014),

$$||f_M^c - f||_\infty \le Ce^{-cM}, \qquad c, C > 0.$$

The discrete Fourier extension

Unfortunately, projection in L^2 is not always possible because often in applications f is only known at finitely many points. Suppose f is only known at the N points $x_1, \ldots, x_N$ given by

$$x_j = \frac{j}{N} - \frac{1}{2} - \frac{1}{2N}, \quad j = 1, 2, \ldots, N.$$

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$$x_j = \frac{j}{N} - \frac{1}{2} - \frac{1}{2N}, \quad j = 1, 2, \dots, N.$$

In order to describe the projection onto S_M^b based on this data, define the discrete inner product $\langle \cdot, \cdot \rangle_N$ as

$$\langle f, g \rangle_N := \frac{1}{N} \sum_{j=1}^{N} f(x_j) \overline{g(x_j)},$$

and let $|| \cdot ||_N$ be the norm inherited from this inner product. Then the approximation by discrete Fourier extension is the function $f_N \in S_M^b$ given by

$$f_M = \mathrm{argmin}\{||q - f||_N : q \in S_M^b\}$$

The minimizer is unique provided $M \leq N$.

Main questions:

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- Need to sample f at N points with $N \geq M$. How many points are necessary to get good (function-independent) convergence?
- Need to choose an extended period $b > 1$. What is a good choice?

A formula for f_N

To define the projection based on the discrete inner product, introduce the orthonormal polynomials with respect to $\langle \cdot, \cdot \rangle_N$. That is, $\varphi_k^N(z)$ be the polynomial of degree k with positive leading coefficient satisfying

$$\frac{1}{N} \sum_{j=1}^{N} \varphi_k^N(z_j) \overline{\varphi_\ell^N(z_j)} = \delta_{k\ell}, \quad z_j = e^{2\pi i x_j / b}.$$

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Then f_M is given by

$$f_M(x) = P_{S_M^b}(f)(x) = \langle f(\cdot), K_M(\cdot, x) \rangle_N,$$

where

$$K_M(x, y) = e^{2\pi i \frac{-(M-1)}{2b}(x-y)} \sum_{\ell=0}^{M-1} \varphi_\ell^N(e^{\frac{2\pi i}{b} x}) \overline{\varphi_\ell^N(e^{\frac{2\pi i}{b} y})}.$$

A formula for the error

The error function is

$$E_{M,N}^{f,b}(x) = (1 - P_{S_M^b})(f)(x) = (1 - P_{S_M^b})(\tilde{f})(x).$$

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Plug in this series for f in the projection formula $P_{S_M^b}$. It gives

$$|E_{M,N}^{f,b}(x)| \leq \sum_{|k|>(M-1)/2} |B_{N,M}^k(x)||a_k|.$$

where

$$B_{N,M}^k(x) := e^{2\pi i x(k+M_0)/b} - \frac{1}{N}\sum_{\ell=0}^{M-1} \varphi_\ell^N(e^{\frac{2\pi i}{b}x}) \sum_{j=1}^{N} e^{2\pi i \frac{k+M_0}{b} x_j} \overline{\varphi_\ell^N(e^{\frac{2\pi i}{b}x_j})},$$

$M_0 = (M-1)/2.$

An estimate on $|B^k_{N,M}(x)|$

Using the Cauchy–Schwarz inequality along with the orthonormality of $\varphi^N_\ell(z)$, we can obtain the simple upper bound

$$\begin{aligned}\left|B^k_{N,M}(x)\right| &= \left| e^{\frac{2\pi i(k+M_0)}{b}x} - \sum_{\ell=0}^{M-1} \varphi^N_\ell(e^{\frac{2\pi i}{b}x}) \langle e^{2\pi i \frac{k+M_0}{b}\cdot}, \varphi^N_\ell(e^{\frac{2\pi i}{b}\cdot}) \rangle_N \right| \\ &\leq 1 + \sum_{l=0}^{M-1} \left| \varphi^N_l(e^{\frac{2\pi i}{b}x}) \right|.\end{aligned}$$

Thus a bound on the orthonormal polynomials gives an upper bound on $\left|B^k_{N,M}(x)\right|$.

Asymtotics of the OP's near the middle

Define the number $\beta \in (0, \frac{1}{2})$ via the equation

$$\cos \frac{2\pi\beta}{b} = \cos(\pi/b) + (1 + \cos(\pi/b)) \tan \left(\frac{\pi M}{2Nb} \right)^2 .$$

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Theorem

Let N and M approach infinity in such a way that the ratio $N/M \geq 1$ remains bounded. Also fix $b > 1$ and fix x such that $|x| < \beta$. Then the orthonormal polynomial $\varphi_M^N(e^{\frac{2\pi i}{b}x})$ satisfies

$$\varphi_M^N(e^{\frac{2\pi i}{b}x}) = \mathcal{O}(1),$$

as $M \to \infty$. Thus the error term $B_{N,M}^k(x)$ satisfies

$$|B_{N,M}^k(x)| = \mathcal{O}(M).$$

These estimates are uniform in x on compact subsets of the interval $(-\beta, \beta)$.

Asymtotics of the OP's near the edges

Theorem

Let N and M approach infinity in such a way that the ratio $N/M = \xi \geq 1$ is bounded. Also fix $b > 1$ and fix x such that $\beta < x < 1/2$. Then the orthonormal polynomial $\varphi_M^N(e^{\frac{2\pi i}{b}x})$ satisfies

$$\varphi_M^N(e^{\pm\frac{2\pi i}{b}x}) = e^{M(L(x)-L(\beta))}\left[F(x)\sin(\pi Nx)+\mathcal{O}(e^{-cM})\right]\left[1+\mathcal{O}(M^{-1})\right],$$

where $F(x)$ is a (complex) bounded analytic function of x and $c > 0$. The error terms are uniform on compact subsets of $(\beta, \frac{1}{2})$. Here $L(x)$ is increasing on $(\beta, 1/2)$.

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But does it really?

A more precise estimate on $B^k_{N,M}$

$B^k_{N,M}(x)$ is written in terms of the Christoffell–Darboux kernel as

$$B^k_{N,M}(x) = e^{2\pi i x(k+M_0)/b} - \frac{1}{N} e^{2\pi i M_0 x/b} \sum_{j=1}^{N} K_M(x, x_j) e^{2\pi i k x_j/b}.$$

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Theorem

Fix $\ell \in \mathbb{Z}_+$, and let $k = \frac{M+1}{2} + \ell$, assuming M is odd. For large enough M, there are $c_\ell > 0$ and $d_\ell > 0$ independent of M such that

$$c_\ell M^\ell e^{ML(\beta)} |\varphi^N_M(e^{\frac{2\pi i}{b}} x)| \leq |B^k_{N,M}(x)| \leq d_\ell M^\ell e^{ML(\beta)} |\varphi^N_M(e^{\frac{2\pi i}{b}} x)|.$$

Properties of $\tilde{L}(x)$

Call the sampling density $N/M := \xi$. Then $L(x) \equiv L(x; b, \xi)$ satisfies the following properties.

- For fixed $b \geq 1$ and $\xi \geq 1$, the function $L(x)$ is constant for $x \in [-\beta, \beta]$, strictly increasing for $x \in [\beta, 1/2]$, and decreasing for $x \in [-1/2, -\beta]$.
- For fixed x and b, $L(x; b, \xi)$ is a decreasing function of $\xi > 1$.
- For any $b > 2$, $L(x; b, \xi) < 0$ for all $\xi \geq 1$ and $x \in [-1/2, 1/2]$.
- For any $1 < b \leq 2$, there exists a sampling density ξ_b such that for any $\xi > \xi_b$, the function $L(x; b, \xi)$ is negative for all $x \in [-1/2, 1/2]$. Conversely, for each $1 \leq \xi < \xi_b$, there exists $x_\xi \in (0, 1/2)$ such that $L(x; b, \xi)$ is positive for all $x \in [-1/2, -x_\xi) \cup (x_\xi, 1/2]$.

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It implies that for each $b \in (1, 2]$, if $\xi < \xi_b$ then $|B_{N,M}^k(x)|$ is exponentially increasing in M for x in a set of positive measure near the end-points of the interval $[-1/2, 1/2]$.

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I would bet some money that

$$\xi_b \equiv \frac{1}{b-1},$$

but I haven't been able to prove it.

An idea of what's going on with these OP's

In the variable $z = e^{2\pi i x/b}$ the orthogonal polynomial $\varphi_M^N(z)$ may be written as the multiple sum (Heine's formula)

$$\varphi_M^N(z) = \frac{1}{D_{M,N}} \sum_{x_1,\dots x_M \in L_N} \prod_{j=1}^{M} (z - e^{2\pi i x_j/b}) \prod_{1 \le j < k \le M} \left| e^{2\pi i x_k/b} - e^{2\pi i x_j/b} \right|^2,$$

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Write it as

$$\frac{1}{D_{M,N}} \sum_{x_1,\dots,x_M=1}^{N} \exp\left[M \int \log(z - e^{2\pi i y/b}) d\nu_{\mathbf{x}}(y) \right] \exp\left[-M^2 H(\nu_{\mathbf{x}}) \right],$$

where

$$H(\nu) = \iint_{x\neq y} \log \frac{1}{|e^{2\pi i x/b} - e^{2\pi i y/b}|} d\nu(x) d\nu(y), \quad \nu_{\mathbf{x}} = \frac{1}{M} \sum_{j=1}^{M} \delta_{x_j}.$$

Since there is a factor M^2 in the exponent, we expect the primary contribution in this integral as $M \to \infty$ to come from a minimizer of the functional $H(\nu)$. We minimize over all Borel measures ν on $[-1/2, 1/2]$ satisfying the following two properties:

1. The measure ν is a probability measure, i.e. $\int_{-1/2}^{1/2} d\nu(x) = 1$.
2. The measure ν does not exceed the limiting density of nodes $x_1, \ldots, x_N$ as $N, M \to \infty$. That is, $0 \le \nu \le \sigma\xi$, where σ is the Lebesgue measure and $\xi := \frac{N}{M}$.

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Then heuristically that for large M,

$$\varphi_M^N(z) \sim \frac{e^{-M^2 E_0}}{D_{M,N}} \exp\left(M \int_{-1/2}^{1/2} \log(z - e^{2\pi i y/b}) d\nu_{\mathrm{eq}}(y) \right),$$

where $E_0 := H(\nu_{\mathrm{eq}})$.

The equilibrium measure is uniquely determined by the *Euler–Lagrange variational conditions*: there exists a *Lagrange multiplier* ℓ such that

$$2\int \log|e^{2\pi ix/b}-e^{2\pi iy/b}|d\nu_{\mathrm{eq}}(y)\begin{cases}\geq \ell & \text{for} \quad x\in \mathrm{supp}\ \nu_{\mathrm{eq}}\\ \leq \ell & \text{for} \quad x\in \mathrm{supp}\ (\xi\sigma-\nu_{\mathrm{eq}}).\end{cases}$$

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If $2\int \log|e^{2\pi ix/b}-e^{2\pi iy/b}|d\nu_{\rm eq}(y)=\ell$ then

$$\varphi_M^N(e^{2\pi ix/b})\sim \frac{e^{-M^2E_0+M\ell/2}}{D_{M,N}},$$

and a similar heuristic shows that

$$D_{M,N}\sim e^{M^2E_0-M\ell/2},$$

so $\varphi_M^N(e^{2\pi ix/b})=\mathcal{O}(1)$ whenever
$2\int \log|e^{2\pi ix/b}-e^{2\pi iy/b}|d\nu_{\rm eq}(y)=\ell$.

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If $2\int \log|e^{2\pi ix/b} - e^{2\pi iy/b}|d\nu_{\mathrm{eq}}(y) = \ell$ then

$$\varphi_M^N(e^{2\pi ix/b}) \sim \frac{e^{-M^2E_0 + M\ell/2}}{D_{M,N}},$$

and a similar heuristic shows that

$$D_{M,N} \sim e^{M^2E_0 - M\ell/2},$$

so $\varphi_M^N(e^{2\pi ix/b}) = \mathcal{O}(1)$ whenever $2\int \log|e^{2\pi ix/b} - e^{2\pi iy/b}|d\nu_{\mathrm{eq}}(y) = \ell$.

It turns out this is the interval $(-\beta, \beta)$

On the other hand, if

$$2\int \log|e^{2\pi ix/b} - e^{2\pi iy/b}| d\nu_{\mathrm{eq}}(y) > \ell,$$

then

$$\varphi_M^N(e^{2\pi ix/b}) \sim e^{M(L(x)-\ell/2)},$$

$$L(x) = \int_{-1/2}^{1/2} \log|e^{2\pi ix/b} - e^{2\pi iy/b}|\, d\nu_{\mathrm{eq}}(y) > \ell/2.$$

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This is the case for $|x| > \beta$. Since $\langle\varphi_M^N, \varphi_M^N\rangle = 1$, it implies $|\varphi_M^N(e^{2\pi ix_j/b})|$ oscillates very regularly in the saturated region, nearly vanishing at each node of L_N, and then growing exponentially large between nodes.

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In the language of discrete orthogonal polynomials this is called a *saturated region*.

How to get rid of that pesky saturated region

If you have the freedom to sample points with a non-constant density (not equally spaced) you should do it! See papers of Adcock et al.

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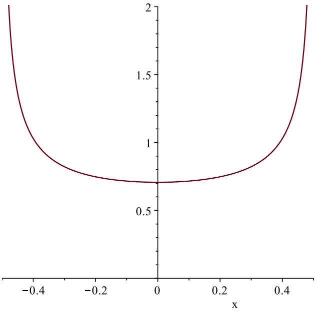
Suppose the N sample points are taken such that the counting measure $\frac{1}{N}\sum_{j=1}^{N}\delta_{x_j}$ converges weakly to some density $\varrho(x)$ as $N \to \infty$. Then the upper constraint on the equilibrium measure is $\xi\varrho(x)$ instead of the constant ξ.

How to get rid of that pesky saturated region

How to choose ϱ? Solve the unconstrained equilibrium problem (just minimize over probability measures). The solution is

$$d\nu_{\mathrm{eq}}^{c}(x) = \frac{\sqrt{2}\cos(\pi x/b)}{b\sqrt{\cos(2\pi x/b) - \cos(\pi/b)}}dx.$$



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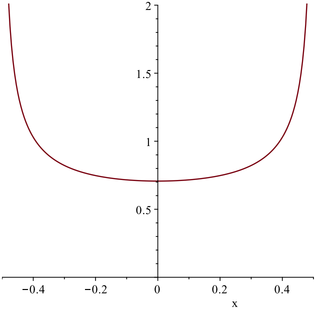

If you choose $\varrho(x)$ at least as big as this density, there will be no saturated region, and the orthogonal polynomials will be $\mathcal{O}(1)$ as $M \to \infty$.

Summary

When implementing the Fourier extension approximation, one must choose a sampling density as well as an extended period $b > 1$. From the point of view of orthogonal polynomial theory, our advice is

- If possible, choose the sample points according to the unconstrained equilibrium measure on previous slide.

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- If you are stuck with equi-spaced sampling taking $b > 2$ will improve some terms in the error.

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When implementing the Fourier extension approximation, one must choose a sampling density as well as an extended period $b > 1$. From the point of view of orthogonal polynomial theory, our advice is

- If possible, choose the sample points according to the unconstrained equilibrium measure on previous slide.
- If you are stuck with equi-spaced sampling taking $b > 2$ will improve some terms in the error.
- If you must take $1 < b < 2$ take the sampling density $\xi = N/M$ to be bigger than $1/(b-1)$. This likewise will improve some terms in the error (close to the endpoints).

Thanks

Thank you much!