Uncertainty for discrete Schrödinger evolutions

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Uncertainty principle

$$f \in L^2(\mathbb{R}), \quad \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2i\pi\xi t} dt$$

Both f and $\hat{f}$ cannot decay fast.

Hardy:

$$f(t) = O(e^{-\pi t^2}),\ \hat{f}(\xi) = O(e{-}\pi\xi^2) \ \Rightarrow \ f(t) = Ae^{-\pi t^2}.$$

Pattern of the proof:

- $\hat{f}$ is an entire function, $|\hat{f}(\xi + i\eta)| < Ce^{\pi\eta^2}$;
- $g(\zeta) := e^{\pi\zeta^2}\hat{f}(\zeta)$ is of order 2 and bounded along $\mathbb{R} \cup i\mathbb{R}$
- next we have to apply convexity arguments

(Phragmen-Lindelöf)

Phragmen-Lindelöf (trigonometric convexity)

Let $\rho > 0$, $A_\rho := \{\zeta, |\arg \zeta| \leq \pi/2\rho\}$; (we will have $\rho = 2$ or 1 only),
$G \in \mathrm{Hol}(A_\rho)$

Definition: H is of order $\rho > 0$ and type $\sigma \geq$ in A_ρ if

$$\limsup_{r\to\infty} \frac{\log\log\{\max_{|\phi|<\pi/\rho} |H(re^{i\phi})|\}}{\log r} \leq \rho;$$

$$\limsup_{r\to\infty} \frac{\log\{\max_{|\phi|<\pi/\rho} |H(re^{i\phi})|\}}{r^\rho} \leq \sigma;$$

Loosely speaking: $|H(z)| < Ce^{|z|^\rho}$

Growth along separate rays - indicator function:

$$h_G(\phi) = \limsup_{r\to\infty} \frac{\log|\phi(re^{i\phi})|}{r^\rho}, \ \sigma = \max_{|\phi|<\pi/\rho} h_G(\phi).$$

Phragmen-Lindelöf (trigonometric convexity)

If H has order ρ in A_ρ its behaviour in the whole angle is defined by the behaviour on the boundary rays.

In particular

$$h_\phi(-\pi/2\rho) + h_\phi(\pi/2\rho) \geq 0$$

If " = " then $h_\phi(\theta) = \alpha \cos \rho\theta + \beta \sin \rho\theta$;

If $\sigma = 0$ and H is bounded on ∂A_ρ then H is bounded in A_ρ.

End of the proof ($\rho = 2$)

$$|\hat{f}(\xi + i\eta)| < Ce^{\pi\eta^2},\ g(\zeta) := e^{\pi\zeta^2}\hat{f}(\zeta) \Rightarrow h_g(\theta) \le \pi \cos^2\theta.$$

$$g \text{ bounded on } \mathbb{R} \cup i\mathbb{R} \ \Rightarrow\ h_g(0), h_g(\pi/2) = 0$$

$\Rightarrow h_g(\theta) = \alpha \sin 2\theta,\ \theta \in (0, \pi/2)$

$$\alpha \sin 2\theta \le \cos^2\theta,\ \ \theta \in (0, \pi/2)\ \Rightarrow \alpha = 0$$

$\Rightarrow g$ has zero growth with respect to order 2...
Now it is easy to complete the proof.

Continuous Schrödinger evolution

$$\partial_t u(t,x) = i\Delta u(t,x),$$

where $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$, Δ is the Laplacian. Solution:

$$u(t,x) = \frac{1}{i\sqrt{2\pi t}} \int_{-\infty}^{\infty} u(0,\xi) e^{i\frac{(x-\xi)^2}{2t}} d\xi$$

$$u(t,x) \underbrace{e^{-i\frac{x^2}{2}t}}_{\text{unimodular}} = \frac{1}{i\sqrt{2\pi t}} \int_{-\infty}^{\infty} u(0,\xi) \underbrace{e^{i\frac{\xi^2}{2t}}}_{\text{unimodular}} e^{i\frac{x\xi}{t}} d\xi$$

$$\partial_t u = i\Delta u, \quad |u(0,x)| + |u(1,x)| \le C\exp(-x^2/4),$$

$$(*) \qquad \Rightarrow \qquad u(0,x) = A\exp(-(1+i)x^2/4)$$

L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega (2006)

Further results

L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega (2006-15)
For any bounded real-valued $V(x,t)$ and any $a > 1/4$

$$\partial_t u = i\Delta u + Vu, \quad |u(0,x)| + |u(1,x)| \le C\exp(-ax^2),$$

$$(**) \qquad \Rightarrow \quad u(t,x) \equiv 0$$

- general elliptic operators
- several dimensions
- non-linear equations
- parabolic equations
- ...

Machinery: Logarithmic convexity for weighted norms + Carleman estimates

Hadamard's three circle theorem (harmonic measure estimates)
Logarithmic convexity of the mean values of harmonic functions over concentric spheres
Elliptic PDE: S. Agmon (1966); Landis and others (1980s), Garofalo and Lin (1987), Brummelhuis (1995)

Schrödinger equation: Escauriaza, Kenig, Ponce, Vega

$$H_R(t) = \|\phi_R(x)u(t,x)\|_2^2, \quad \phi_R(x) = \exp(\gamma|x + Rt(1-t)|^2)$$

$$\partial_t^2 \log H_R(t) \geq -R^2(4\gamma)^{-1}$$

$$\exp(-R^2(16\gamma)^{-1})H_R(1/2) \leq H_R(0)^{1/2}H_R(1)^{1/2} = H(0)^{1/2}H(1)^{1/2}$$

Let $R \to \infty$ and get a contrudiction when $\gamma > \gamma_0$.

Discrete Schrödinger evolutions

Equation

$$\partial_t u = i(\Delta_d u + Vu),$$

where $u : \mathbb{R}_+ \times \mathbb{Z} \to \mathbb{C}$ and Δ_d is the discrete Laplacian, that is, for a complex valued function $f : \mathbb{Z} \to \mathbb{C}$,

$$\Delta_d f(n) := f(n+1) + f(n-1) - 2f(n).$$

We assume that the potential $V = V(t, n)$ is a real-valued bounded function.

Uniqueness ?

$$|u(0, n)| + |u(1, n)| \leq Cm(n) \quad \Rightarrow \quad u \equiv 0.$$

Discrete potential theory: old and new

Chang and Yau, 1997, 2000 (Calculation/estimates of the discrete heat kernels)

Three spheres theorem and logarithmic convexity for weighted norms of discrete harmonic functions:
Gaudi and Malinnikova (Compt. Methods and Function Th, 2014)
Lippner and Mangoubi (arXiv 2013, to appear in Duke Math. J.)

Heisenberg's uncertainty, interpretation for discrete Schrödinger evolution: Fernández-Bertolin (arXiv 2014, to appear in AHCA)

Our goals

- Model cases: precise results and guesses for more general cases.
- For more general cases: prove the logarithmic convexity of the norms

$$H(t) = \|\psi(n)u(t,n)\|_2$$

for appropriate ψ and thus obtain uniqueness results

Free discrete Schrödinger

Proposition

Let $\partial_t u = i\Delta_d u$, *and*

$$|u(0,n)|,\ |u(1,n)| \le C\frac{1}{\sqrt{|n|}}\left(\frac{e}{2|n|}\right)^{|n|} \sim J_n(1) \sim 2^{-n}(n!)^{-1}.$$

Then $u(t,n) = Ai^{-n}e^{-2it}J_n(1-2t)$ *for all* $n \in \mathbb{Z}$ *and* $0 \le t \le 1$, *for some constant* A.

Comment: Another speed of decay !

Compactly supported potentials

Theorem

Let $u : \mathbb{R}_+ \times \mathbb{Z} \to \mathbb{C}$*,*

$$\partial_t u = i(\Delta_d u + Vu),$$

where the potential V *does not depend on time and also* $V(n) \neq 0$ *just for a finite number of* n*'s. If, for some* $\varepsilon > 0$*,*

$$|u(t,n)| \leq C \left(\frac{e}{(2+\varepsilon)n} \right)^n, \quad n > 0, \; t \in \{0; 1\},$$

then $u = 0$*.*

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then $u = 0$*.*

Jost solutions; one-sided estimates, entire functions...

General potentials

Theorem

If u is a strong solution of

$$\partial_t u = i(\Delta_d u + Vu)$$

where $V(t,n)$ is a real-valued bounded function,

$$\|(1+|n|)^{\gamma(1+|n|)}u(0,n)\|_2, \|(1+|n|)^{\gamma(1+|n|)}u(1,n)\|_2 < +\infty,$$

then for $\gamma > \gamma_0$, then $u \equiv 0$.

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Comments: 1. One can take$\gamma_0 = (3+\sqrt{3})/2$ but this cannot be the best.
2. Dream weight: $m(n) = J_n(1) = \psi^{-1}(n)$. (free Schrödinger, heat kernel)
3. Strategy of the proof: improvement of improvement of improvement

Step 1: First energy estimate

Weight function:

$$\psi_\alpha(t) = \{\psi_\alpha(t,n)\}_{n\in\mathbb{Z}} = \{(1+|n|)^{\alpha|n|/(1+t)}\}_{n\in\mathbb{Z}}$$

$$H(t) := \|\{\psi_\alpha(t,n)u(t,n)\}\|_{l^2}^2.$$

Proposition

Let $V = V_1 + iV_2$, with $V_1, V_2 : [0,T]\times\mathbb{Z} \to \mathbb{R}$ and V_2 bounded and $F : [0,T]\times\mathbb{Z}\to\mathbb{C}$ bounded,

$$\partial_t u(t,n) = i(\Delta u(t,n) + V(t,n)u + F(t,n)).$$

Assume that $\{\psi_\alpha(0,n)u(0,n)\} \in \ell^2(\mathbb{Z})$ for some $\alpha \in (0,1]$. Then

$$H(T) \le e^{CT}\left(H(0) + \int_0^T \|\psi_\alpha(s,n)F(s,n)\|_2^2\, ds\right)$$

Step 2: Estimates with an auxiliary weight

Proposition

Let $\gamma > 0$. Assume that u is a strong solution of

$$\partial_t u = i(\Delta_d u + Vu)$$

where the potential V is a bounded real-valued function. Let also

$$\|(1+|n|)^{\gamma(1+|n|)} u(t,n)\|_2 < +\infty, \quad t \in \{0;1\}.$$

Then, for all $t \in [0,1]$, $\|(1+|n|)^{\gamma(1+|n|)} u(t,n)\|_2 < +\infty$.

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weight and logarithmic convexity

$$\psi(n) = e^{\kappa_b(n)}, \; \kappa_b(n) = \gamma(1+|n|)\ln^b\big(1+|n|\big),$$

where $1/2 < b < 1$, then $b \to 1$.

Step 3: Final convexity estimates with a parameter. Carleman estimate

$\psi(t, n) = e^{\kappa(t,n)}$, where

$$\kappa(t, n) = \gamma(|n| + C_0 + Rt(1-t)) \ln(|n| + C_0 + Rt(1-t)).$$

C_0 being large enough.
As before $\psi(t, n) = e^{\kappa(t,n)}$, and $H(t) = \|u(t, n)\psi(t, n)\|_2^2$
Log convexity:

Lemma

For every $\gamma > (3 + \sqrt{3})/2$ there exists $C(\gamma)$ such that for $C_0 > C(\gamma)$ and $R(t) = C_0 + R_0 t(1-t)$ we have

$$\partial_t^2(\log H(t)) \geq -\frac{4\gamma}{2\gamma - 3} R_0 \log R_0 - C_1 R_0 - C_2,$$

where C_1 and C_2 depend on γ and $\|V\|_\infty$ only.

Compactly supported potentials

Theorem

Let $u : \mathbb{R}_+ \times \mathbb{Z} \to \mathbb{C}$,

$$\partial_t u = i(\Delta_d u + Vu),$$

where the potential V *does not depend on time and also* $V(n) = 0$ *for* $|n| > N$. *If, for some* $\varepsilon > 0$,

$$|u(t,n)| \leq C \left(\frac{e}{(2+\varepsilon)n} \right)^n, \quad n > 0, \; t = 0, 1,$$

then $u = 0$.

Compactly supported potentials

Jost solutions: - eigenfunctions of Δ_d

$$\Delta_d e^{\pm}(\theta) = \lambda(\theta) e^{\pm}(\theta), \quad \lambda(\theta) = 2 - \theta - \theta^{-1};$$

$$e^{\pm}(\theta, n) = \theta^n, \text{ for } \pm n > N.$$

$$e^{-}(\theta, n) = a(\theta) e^{+}(\theta, n) + b(\theta) e^{+}(\theta^{-1}, n).$$

Fact:
$a(\theta)$, $b(\theta)$, $e^{\pm}(\theta, n)$ - all are rational functions.

Compactly supported potentials

Spectral Fourier transform:

$\Phi(t,\theta)=\sum_{n=-\infty}^{\infty}e^{-}(\theta,n)u(n,t)$ well-defined for $\theta\in\mathbb{T}$, $t\geq 0$.

$$i\partial_t\Phi(t,\theta)=\sum_{n=-\infty}^{\infty}e^{-}(\theta,n)\Delta_d u(n,t)=(2-\theta-\theta^{-1})\Phi(t,\theta)$$

Therefore

$$\Phi(1,\theta)=e^{-i(2-\theta-\theta^{-1})}\Phi(0,\theta),\ \ \theta\in\mathbb{T}.$$

$u(n,1),u(n,0)$ decay $\Rightarrow$ this relation can be extended to the whole $\mathbb{C}$!

Compactly supported potentials

Behaviour at ∞:

$$\Phi(\nu,\theta) = \sum_{n=-\infty}^{0} e^{-}(\theta,n)u(n,\nu) + a(\theta)\sum_{n=1}^{\infty} e^{+}(\theta,n)u(n,\nu)$$

$$+\, b(\theta)\sum_{n=1}^{\infty} e^{+}(\theta,n)u(n,\nu) = A_\nu(\theta) + B_\nu(\theta) + C_\nu(\theta),\ \ \nu = 0,1.$$

Only B contains infinitely large positive powers of $\theta \Rightarrow$

$$\limsup_{r\to\infty} \frac{\log|A(re^{i\phi})|}{r},\ \ \limsup_{r\to\infty} \frac{\log|A(re^{i\phi})|}{r} = 0,\ \ \phi\in[0,2\pi].$$

Compactly supported potentials

Estimate of the solutions $u(\nu, n)$ yields estimates of $B_\nu(\theta)$:

$$|u(\nu, n)| \le Ce^n(2+\varepsilon n)^{-n}, \ \nu = 0, 1 \Rightarrow$$
$$\limsup_{r\to\infty} \frac{\log|B_\nu(re^{i\phi})|}{r} < 1/(2+\varepsilon), \ \phi \in [0, 2\pi].$$

Phragmen-Lindelöf $\Rightarrow$

$$\liminf_{r\to\infty} \frac{\log|B_\nu(re^{i\phi})|}{r} > -1/(2+\varepsilon), \ \phi \in [0, 2\pi];$$

and

$$\left|\limsup_{r\to\infty} \frac{\log|\Phi(\nu, re^{i\phi})|}{r}\right| < 1/(2+\varepsilon), \ \phi \in [0, 2\pi]$$

Finallly

Take $\phi = \pi/2$, $re^{i\phi} = iy$:

$$\underbrace{\limsup_{y\to\infty} \frac{\log|\Phi(0,iy)|}{y}}_{|\cdot|<1/(2+\varepsilon)} = \underbrace{\lim_{y\to\infty} \frac{\log|e^{-i(y-1/y)}|}{y}}_{=1} + \underbrace{\limsup_{y\to\infty} \frac{\log|\Phi(1,iy)|}{y}}_{|\cdot|<1/(2+\varepsilon)}.$$