Local regularity, multifractal analysis and boundary behavior of harmonic functions

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- Oscillation integral and the law of the iterated logarithm

Local regularity

Let $f : \mathbf{R} \to \mathbf{R}$ and $\alpha > 0$, we say that $f \in C^\alpha(x_0)$ if there exists a polynomial P of degree less than α such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha, \ |x - x_0| < 1.$$

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Jaffard (1996) computed $h_R(x)$ explicitly,
$1/2 \leq h_R(x) \leq 3/2$ depends on the rate of rational approximation.

Wavelet transform

Local regularity can be measured by the decay of the wavelet transform

$$W_f(a,b) = \frac{1}{a}\int_{\mathbf{R}} f(t)\psi(a^{-1}(t-b))dt,$$

where ψ is a "wavelet-function", ψ is smooth enough and

$$\int \psi(t)dt = 0.$$

Roughly speaking, $f \in C^\alpha(x_0)$ iff

$$|W_f(a,b)| \le Ca^\alpha(1 + a^{-1}|b - x_0|)^\alpha.$$

Spectrum of singularities

Let

$$E_f(\beta) = \{x \in \mathbf{R} : h_f(x) = \beta\}$$

$$d_f(\beta) = \dim_H(E_f(\beta)),$$

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EXAMPLE:

$$d_R(\beta) = ?$$

Local dimension of a measure

Let μ be a positive measure on $\mathbf{R}^{m-1}$, we define the (lower) local dimension of μ at x_0 as

$$h_\mu(x_0) = \liminf_{r \to 0+} \frac{\log \mu(B(r, x_0))}{\log r}.$$

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We will instead work with the harmonic extension $u = P * \mu$, we define

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Exercise

The following estimate holds $\dim_H F_\gamma(u) \le m - 1 - \gamma$, *and it is sharp.*

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Theorem (K.S. Eikrem, M., 2012; F. Bayart, Y. Heurteaux, 2013))

(i) Let u be a positive harmonic function in $\mathbb{R}^m_+$, we define

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(ii) There exists a positive function u such that $u(y,t) \le v(t)$ and $\mathcal{H}_\lambda(E_v(u)) > 0$, where

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For a typical (Baire category) positive measure the set of given growth has exactly this "Hausdorff dimension".

Classes of harmonic functions of controlled growth

Let $v(t)$, $t > 0$, be a positive increasing continuous function and assume that $\lim_{t\to 0+} v(t) = +\infty$. We define

$$k_v = \{u : \mathbf{R}^m_+ \to \mathbf{R}, \Delta u = 0, u(y,t) \le Kv(t)\},$$

and

$$h_v = \{u : \mathbf{R}^m_+ \to \mathbf{R}, \Delta u = 0, |u(y,t)| \le Kv(|t|)\}.$$

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Similar spaces can be considered in the unit disc (ball).

For any v there exists $u \in h_v$ such that $u(ry) \to \infty$ for a.e. $y \in \mathbf{S}$ (N. Lusin, I. Privalov; J.-P. Kahane, Y. Katsnelson). This behavior is very different of the one we have seen for positive harmonic functions.

Some examples and constructions

Our main examples of weights are $v_1(t) = t^{-\alpha}$ and $v_2(t) = |\log t|^{\beta}$. Examples of corresponding functions in the unit disc:

$$u(z) = \Re \sum_n n^{\alpha-1} z^n, \qquad u(z) = \Re \sum_n 2^{n\alpha} z^{2^n}$$

$$u(z) = \Re \sum_n n^{\beta-1} z^{2^n}, \qquad u(z) = \Re \sum_n 2^{\beta n} z^{2^{2^n}}$$

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Another way to produce (regular) examples is to work with generalized Cantor sets on $\mathbf{S}$. However there are much less regularly behaving functions in h_v.

Sets of extremal growth

Let $u \in k_v$, we consider

$$E_v^+(u) = \{y \in \mathbf{S} : \liminf_{t \to 0} \frac{u(y,t)}{v(t)} > 0\}.$$

$E_v^+(u)$ consists of the end points of vertical rays along which u grows as v. Similarly

$$E_v^-(u) = \{y \in \mathbf{S} : \limsup_{t \to 0} \frac{u(y,t)}{v(t)} < 0\}.$$

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Theorem (Borichev, Lyubarskii, Thomas, M., 2009)

Let $m = 2$. Assume that for any $\omega > 0$, $\lambda(t) = o(t|\log t|^\omega), (t \to 0)$. Then for each $u \in k_{\log}$ we have $\mathcal{H}_\lambda(E^+(u)) = \mathcal{H}_\lambda(E^-(u)) = 0$.

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A similar result is true for any $m \geq 2$ and any v satisfying the doubling condition $v(t) \leq Cv(2t)$ (Eikrem, M., 2012).

Sharpness of results

Theorem (Eikrem, M., 2012)

For any $\alpha > 0$ there exists $u \in h_v$ such that $\mathcal{H}_\lambda(E_v^\pm(u)) > 0$ for $\lambda(t) = t^{m-1}v(t)^\alpha$.

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Problem

Estimate the size of the sets $E_w^{\pm}(u)$ when $u \in h_v$ and w is "smaller" than v.

Makarov's law of the iterated logarithm

Consider the following function

$$u(z) = \Re \sum_n z^{2^n}$$

This is a sum of independent random variables, it satisfies the law of the iterated logarithm.
Makarov: Suppose that $u(z) \in \mathcal{B}$ (Bloch space), i.e.

$$|\nabla u(z)| \leq C(1-|z|)^{-1}, \ \Delta u = 0,$$

then

$$\limsup_{r\to 1-} \frac{|u(re^{i\phi})|}{\sqrt{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}}} \leq C$$

for a.e. ϕ.

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We now turn to our second examples

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To measure such oscillation of functions in $h_{\log}$ we introduce the weighted integral

$$I_u(R,\phi) = \int_{1/2}^{R} \frac{u(re^{i\phi})}{(1-r)\left(\log \frac{1}{1-r}\right)^2} dr, \quad R \in (0,1),\ \phi \in (-\pi,\pi).$$

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Clearly $I_u(R,\phi) \le I_{|u|}(R,\phi) \le C \log|\log(1-R)|$. We show that I_u grows much slower.

A law of the iterated logarithm

Theorem (Lyubarskii, M., 2012)

There exists K such that if u is a harmonic function in $\mathbf{D}$ satisfying

$$|u(z)| \le \log \frac{e}{1-|z|},$$

then

$$\limsup_{R \nearrow 1} I_u(R,\phi) \left(\log\log \frac{1}{1-R} \log_4 \frac{1}{1-R} \right)^{-1/2} \le K$$

for almost every $\phi \in (-\pi, \pi]$.

Premeasures and martingales

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$$g_n = \sum_{I \in |I| = 2\pi 2^{-n}} \mathbf{1}_I \frac{\mu(I)}{|I|},$$

where I are dyadic subintervals of $(-\pi, \pi)$. We define

$$d_j = 2^{-j}(g_j - g_{j-1}), \quad \text{and} \quad f_n = \sum_{j=1}^{n} d_j.$$

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Then the martingale $\{f_n\}$ obeys the Kolmogorov's law of the iterated logarithm. An approximation of the Poisson kernel by the box kernel suggests that $I_u(1 - 2^{-2^n}, \cdot)$ can be approximated by f_n above.

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We use decomposition into atoms and careful error estimates instead, however our argument is based on the Korenblum premeasures.

Besov Spaces $B^{-s,\infty}_{\infty}$

For $s > 0$ we have $T \in B^{-s,\infty}_{\infty}$ if and only if $\|P_y * T\|_{\infty} \leq Cy^{-s}$, $y < 1$. For $s = 0$ the corresponding Besov space $B^{0,\infty}_{\infty} = \mathbf{B}$ is the Bloch space and $T \in \mathbf{B}$ if and only if $\|\nabla(P_y * T)\|_{\infty} \leq Cy^{-1}$.

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Wavelet transform: $T \in B_{\infty}^{-s,\infty}$ with $s \geq 0$ if and only if $W_T(a, b) \leq Ca^{-s}$ (there is a freedom to choose the wavelet function you like).

Spaces of boundary distributions

We define the space of distributions

$$D_\infty(v) = \{T : |P_y * T| \le C(T)v(y)\}$$

(boundary values of functions in h_v).

Theorem (Eikrem, Mozolyako,M.,2014)

Let T be a distribution of finite order s that admits convolutions with the Poisson kernel and let W be the wavelet-transform with some smooth enough wavelet ψ. Then $T \in D_\infty(v)$ if and only if $\|W_T(a,\cdot)\|_\infty \le C(T)v(a)$.

Oscillation for general weights

As above, we describe the oscillation by the following weighted average

$$I_{u,v}(x,s) = \int_s^1 u(x,y)\,d(v^{-1}(y)).$$

Theorem (Eikrem, Mozolyako, M., 2014)

Let $u \in h_v$ then

$$\limsup_{y\to 0} \frac{|I_u(x,s)|}{\sqrt{\log v(s) \log\log\log v(s)}} \le C$$

for almost every $x \in \mathbf{R}^{m-1}$.

Open problems

1. The last result provides some weights w ($w << v$) for which $\mathcal{H}^{m-1}(E_w(u)) = 0$ when $u \in h_v$ but we don't know exact description.

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3. Construct an example of a pair of weights v, w and $u \in h_v$ such that $\mathcal{H}^{m-1}(E_w(u)) > 0$.

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3. Construct an example of a pair of weights v, w and $u \in h_v$ such that $\mathcal{H}^{m-1}(E_w(u)) > 0$.
4. Describe (typical) local regularity of a premesure that satisfies a one-sided estimate $\mu(I) \leq |I| v(|I|)$.

Answer

EXAMPLE:

$$d_R(\beta) = \begin{cases} 4\beta - 2, & 1/2 \le \beta \le 3/4 \\ 0, & \beta = 3/2 \\ -\infty & \text{otherwise} \end{cases}$$

Thank you

Thank you for your attention

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