On equilibrium problems on the real axis. Applications

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Motivation

Many applications of Equilibrium Problems in the External fields:

- Asymptotics of orthogonal polynomials
  - With respect to exponential weights
  - With respect to general varying weights (connection with multipoint rational approximation)
- Asymptotics of Heine-Stieltjes polynomials
- Limit mean density of eigenvalues of Random Matrices
- Continuum limit of Toda lattice and Soliton Theory (KdV)

Equilibrium Problem on a compact set

- K : compact subset of $\mathbb{R}$.
- $t > 0$, $M_t(K)$: measures σ supported on K such that $\sigma(K) = t$.

Under quite mild conditions on K, there exists a unique measure (Equilibrium or Robin measure) $\mu_{eq} = \mu_{eq,t}$, $supp\,\mu_{eq} \subset K$, minimizing the

Energy

$$I(\sigma) = -\int\int \log|x-z|\,d\sigma(x)d\sigma(z)\,, \quad \sigma \in M_t(K)\,.$$

- $supp\,\mu_{eq} = K$.
- $V^{\mu_{eq}}(z) = -\int \log|x-z|d\mu_{eq}(x) = c_t = const$, $z \in K$.
- $\min_{z\in K} V^{\mu_{eq}}(z) = \max_{\sigma\in M_t(K)} \left(\min_{z\in supp\,\sigma} V(\sigma;z)\right)$.

Equilibrium Problem on a compact set

The simplest example: $K = [a, b] \subset \mathbb{R}\,,\ t = 1$

Equilibrium measure

$$d\mu_{eq}(x) = \frac{1}{\pi}\frac{dx}{\sqrt{(x-a)(b-x)}}\,,\ x \in (a, b)$$

$$V^{\mu_{eq}}(z) = \log\left(\frac{4}{b-a}\right)\,,\ x \in (a, b)$$

Equilibrium Problem on a compact set

A bit more involved example:
$K = [a, b] \cup [c, d] \subset \mathbb{R}\,,\; b < c\,,\; t = 1$

Equilibrium measure

$$d\mu(x) = \frac{1}{\pi}\frac{(x-\xi)\,dx}{\sqrt{|(x-a)(x-b)(x-c)(x-d)|}}\,,\; x \in (a,b)\cup(c,d)$$

$$\xi \in (b,c)\,,\; \int_b^c \frac{(x-\xi)\,dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = 0$$

Equilibrium Problems in the presence of external fields

- Σ a closed subset of $\mathbb{C}$ (possibly unbounded)
- φ an "admissible" external field (Saff-Totik, 1997), $\omega(z) = e^{-\varphi(z)}$ (weight).
  In particular, for an unbounded Σ, and certain $t > 0$, it means:

$$\lim_{|z|\to\infty,\, z\in\Sigma} (\varphi(z) - t \log|z|) = +\infty.$$

Then, there exists a unique measure $\mu_\varphi = \mu_{\varphi,t} \in M_t(\Sigma)$ minimizing the

Weighted Energy

$$\begin{aligned} I_\varphi(\sigma) &= -\int\int \log\left(|x-z|\omega(x)\omega(z)\right)\, d\sigma(x)\, d\sigma(z) \\ &= -\int\int \log|x-z|\, d\sigma(x)\, d\sigma(z) + 2\int \varphi(x)\, d\sigma(x) \end{aligned}$$

Equilibrium Problems in the presence of external fields

$S_\varphi = S_{\varphi,t} == \text{supp } \mu_\varphi$ is a compact subset of Σ.

Total ("chemical") potential

$$W^{\mu_\varphi}(z) = V^{\mu_\varphi}(z) + \varphi(z) \begin{cases} = F_\omega = F_{\omega,t}, & z \in S_\varphi \\ \geq F_\omega, & z \in \Sigma \end{cases}$$

S_φ maximizes the

F-functional (Mhaskar-Saff, Saff-Totik)

$$F(K) = t \log \text{cap}\,(K) - \int \varphi(x) d\mu_{eq,K}(x)$$

among all the compact subsets K of Σ.

Suppose that $\Sigma \subset \mathbb{R}$. Then:

- φ convex $\Longrightarrow$ S_φ is an interval.
- φ real analytic $\Longrightarrow$ S_φ is comprised by a finite union of intervals.

But... in general, finding the support S_φ is a difficult task!

Simple examples

- $\varphi(x) = x^2$, $t = 1$. φ is convex and symmetric $\Longrightarrow S_\varphi$ maximize $F(K)$, with $K = [-a, a]$, $a > 0 \Longrightarrow S_\varphi = [-1, 1]$ and
$$\mu'_\varphi(x) = \frac{2}{\pi}\sqrt{1 - x^2}$$
- $\varphi(x) = \frac{2}{3}x^4$, $t = 1$. $\Longrightarrow S_\varphi = [-1, 1]$ and
$$\mu'_\varphi(x) = \frac{4}{3\pi}(1 + 2x^2)\sqrt{1 - x^2}$$

Rational External Fields

$$\varphi(x) = P(x) + \sum_{j=1}^{q} \alpha_j \log|x - z_j|, \; q \geq 1, \; z_j \in \mathbb{C} \setminus \mathbb{R}, \; \alpha_j \in \mathbb{R},$$

$$P(x) = \frac{x^{2p}}{2p} + \sum_{j=1}^{2p-1} c_j \, x^j, \;\; p > 0 \text{ or } P \equiv 0,$$

$$\varphi'(x) = P'(x) + \sum_{j=1}^{q} \frac{\alpha_j \, (x - \operatorname{Re} z_j)}{(x - z_j)(x - \overline{z_j})} = \frac{E(x)}{D(x)},$$

$$D(x) = \prod_{j=1}^{q} (x - z_j)(x - \overline{z_j}).$$

Rational External Fields

- If $p > 0$ (φ has a polynomial part)
  $\Longrightarrow$ φ is admissible for any $t > 0$

- If $p = 0$ (φ is "purely rational") $\Longrightarrow$ φ is weaker
  $\Longrightarrow$ φ is admissible only for $t \in (0, T)$,

$$T = \sum_{j=1}^{q} \alpha_j$$

Rational External Fields

Equilibrium measure

- Size of the measure, "time" or "temperature"

$$M_t(\mathbb{R}) = \{\sigma \ : \sigma(\mathbb{R}) = t > 0\}$$

- Support of the equilibrium measure

$$S_t = S_{\varphi,t} = \operatorname{supp} \mu_t = \bigcup_{j=1}^{k} [a_{2j-1}, a_{2j}]\,,\ 1 \leq k \leq p+q$$

- Density of the equilibrium measure

$$\mu_t'(z) = \frac{1}{\pi}\,\frac{B(z)\,\sqrt{A(z)}}{D(z)}\,,\quad A(z) = \prod_{j=1}^{2k}(z-a_j)$$

First tool: Algebraic equation for the Cauchy Transform

$$((-\widehat{\mu}_t(z) + \varphi'(z))^2 = R(z) = \frac{B(z)^2\, A(z)}{D(z)^2}\,,\; z \in \mathbb{C} \setminus S_t\,,$$

$$\widehat{\mu}_t(z) = \int \frac{d\mu_t(y)}{z-y}$$

$$A(z) = \prod_{j=1}^{2k} (z - a_j)\,, B(z) = \prod_{j=1}^{2(p+q)-k-1} (z - b_j),$$

$$a_1, \ldots, a_{2k} \in \mathbb{R}\,,\; 1 \leq k \leq p+q.$$

$$D(x) = \prod_{j=1}^{q} (x - z_j)(x - \overline{z_j})$$

Rational external fields

Second tool: A dynamical viewpoint (Buyarov-Rakhmanov, 1999)

- Let $t \in (0, +\infty)$. Except for a few values of t, μ_t and its support S_t depend analytically on t
- $\frac{d\mu_t}{dt}|_{t=t_0} = \omega_{t_0}$,

  ω_{t_0}: Robin measure (equilibrium measure in absence of external field) of the compact set S_{t_0}.

$\Longrightarrow$ A dynamical system for zeros of A and B: endpoints of S_t and other zeros of the density!!!

Rational External Fields

Dynamical system for zeros of A and B

$$\dot{a_j} = \frac{\partial a_j}{\partial t} = -\frac{2D(a_j)F(a_j)}{\prod_{k\neq j}(a_j - a_k)\, B(a_j)}\,,\; j = 1,\ldots,2k\,,$$

$$\dot{b_j} = \frac{\partial b_j}{\partial t} = -\frac{D(b_j)F(b_j)}{\prod_{k\neq j}(b_j - b_k)\, A(b_j)}\,,\; j = 1,\ldots,2(p+q)-k-1\,,$$

$$F \in \mathbb{P}_{k-1}\; : \int_{a_{2j}}^{a_{2j+1}} \frac{F(x)}{\sqrt{A(x)}}\,,dx \;=\; 0\,,\; j = 1,\ldots,k-1\,.$$

$$D(x) = \prod_{j=1}^{q} (x - z_j)(x - \overline{z_j})$$

Rational external fields

Singularities: collisions/bifurcations of zeros of A and/or B

- **Singularity of type I:** at a time $t = T$ a real zero b of B (a double zero of R_t) splits into two simple zeros $a_- < a_+$, and the interval $[a_-, a_+]$ becomes part of S_t (*birth of a cut*). Phase transition: the number of cuts increases.
- **Singularity of type II:** at a time $t = T$ two simple zeros a_{2s} and a_{2s+1} of A (simple zeros of R_t) collide (*fusion of two cuts* or *closing of a gap*). Phase transition: the number of cuts decreases.
- **Singularity of type III:** at a time $t = T$ a pair of complex conjugate zeros b and $\overline{b}$ of B (double zeros of R_t) collide with a simple zero a of A, so that $\lambda'_T(x) = \mathcal{O}(|x-a|^{5/2})$ as $x \to a$. No phase transition occurs: the number of cuts remains unchanged.

Particular case: Polynomial external fields

General Polynomial External Field

$$\varphi(x) = \frac{x^{2p}}{2p} + \sum_{j=1}^{2p-1} t_j\, x^j, \quad t_j \in \mathbb{R},$$

Bleher, Eynard, Its, Kuijlaars, McLaughlin...

A. Martínez Finkelshtein, RO, E. A. Rakhmanov (CMP, 2015)

A suitably combined use of two ingredients $\Longrightarrow$
Full description of dynamics in the Quartic case:

$$\varphi(x) = \frac{x^4}{4} + t_3 x^3 + t_2 x^2 + t_1 x\,.$$

In particular: Two-cut is possible iff φ is a "sufficiently non-convex" external field:
Simple geometrical characterization in terms of the relative position of critical points of φ

Motivation: Random Matrix Models

G. S. Krishnaswami (2006):
1-matrix model whose action is given by

$$V(M) = tr\ \left(M^4 - \log(v + M^2)\right),$$

Computable toy-model for the gluon correlations in a baryon background. Generalized Gauss-Penner model:
$\varphi(x) = ax^4 + bx^2 - c\ln|x|$, extending the classical Gauss-Pener model: $\varphi(x) = x^2 - k\ln|x|$.
$\Longrightarrow$

$$\varphi(x) = x^4 - \log(x^2 + v)\,,\ v > 0\,.$$

A particular case: Generalized Gauss-Penner model

RO, J. Sánchez Lara (JMAA, 2015)

$$\varphi(x) = \alpha x^4 + \beta x^2 + \gamma \log(x^2 + v)\,,\ \beta, \gamma \in \mathbb{R}\,,\ \alpha, v > 0\,,$$

$$\Downarrow$$

$$\phi(x) = 2\varphi(\sqrt{x}) = 2\alpha x^2 + 2\beta x + 2\gamma \log(x + v)\,,\ x \in [0, +\infty)\,.$$

$$\Downarrow$$

Simplified model in $[0, +\infty)$

$$\phi(x) = \frac{1}{2}x^2 + \beta x + \gamma \log(x + 1)\,,\ x \in [0, +\infty)\,.$$

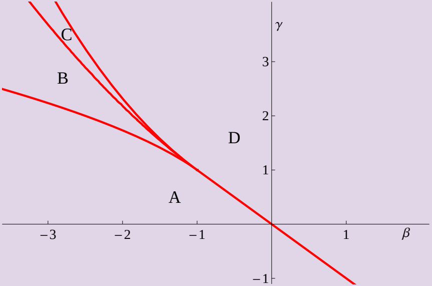
A particular case: Generalized Gauss-Penner model

Dynamics of the support wrt to t. Main questions

- For which values of (β, γ) is it feasible the two-cut case, that is:
  When there exists some time interval (T_1, T_2), $0 \leq T_1 < T_2$, such that S_t is comprised by two disjoint intervals for $t \in (T_1, T_2)$?
  In this case, the support of the original equilibrium measure in the presence of φ is comprised by three disjoint intervals (three-cut case).
- For which values of (β, γ), $S_t = [0, a_1(t)]$, for any $t > 0$ where $a_1(t)$ is an increasing function of t? In this case, we have the one-cut case for the original equilibrium measure for every $t > 0$.

A particular case: Generalized Gauss-Penner model

First questions: When two-cut is feasible?



- Region D. 1 phase: $[0, a_1]$, $\forall t > 0$.
- Region A. 2 phases for S_t: $[a_2, a_3]$, $0 < a_2 \longrightarrow [0, a_3]$
- Region B. 3 phases: $[a_2, a_3] \longrightarrow [0, a_1] \cup [a_2, a_3] \longrightarrow [0, a_3]$
- Region C. 3 phases: $[0, a_1] \longrightarrow [0, a_1] \cup [a_2, a_3] \longrightarrow [0, a_1]$

"Purely" Rational External Fields: without polynomial part.

Motivation: Generalized Heine-Stieltjes Polynomials

$$E_\varphi(\zeta_1, \ldots, \zeta_n) = \sum_{i<j} \log \frac{1}{|\zeta_i - \zeta_j|} + \sum_{i=1}^{n} \varphi(\zeta_i)$$

$$= \frac{1}{2} \left(\sum_{i \neq j} \log \frac{1}{|\zeta_i - \zeta_j|} + 2 \sum_{i=1} \varphi(\zeta_i) \right) .$$

If for each n, $(\zeta_1^*, \ldots, \zeta_n^*)$ is minimal (Weighted Fekete Points), then

$$\nu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\zeta_i^*} \xrightarrow{*} \mu_t = \mu_t(\varphi) .$$

"Purely" Rational External Fields

Motivation: Generalized Heine-Stieltjes Polynomials

In particular, when

$$\varphi(x) = \sum_{j=1}^{q} \alpha_j \log|x - z_j|\,,\; q \geq 1\,,\; z_j \in \mathbb{C} \setminus \mathbb{R}\,,\; \alpha_j \in \mathbb{R}\,,$$

Then, $y(x) = y_n(x) = \prod_{i=1}^{n} (x - \zeta_i^*)$ (Heine-Stieltjes Polynomials) satisfy a linear ODE:

$$A_n y'' + B_n y' + C_n y = 0\,,$$

A_n, B_n, C_n polynomials (generalized Lamé equation).

“Purely” Rational External Fields

Motivation: Generalized Multi-Penner models in Random Matrix theory

Action given by

$$W(M) = \sum_{j=1}^{N} \mu_j \log(M - q_i)\,,$$

Interest in Gauge Theory, as well as in Toda systems (R. Dijkgraaf, C. Vafa (2009), T. Eguchi (2010)).

Example of "purely" rational external fields: External Fields created by a couple of attractors

(RO–J. Sánchez, preprint):

$$\varphi(x) = \log|x - z_1| + \gamma \log|x - z_2|,$$

$$z_1 = -1 + \beta_1 i, \; z_2 = 1 + \beta_2 i, \; \beta_1, \beta_2 > 0, \; \gamma > 0$$

$$\varphi(x) = \frac{1}{2}\left(\log((x+1)^2 + \beta_1^2) + \gamma \log((x-1)^2 + \beta_2^2)\right)$$

is not convex but is real analytic $\Longrightarrow$ the support S_t, $t \in (0, 1+\gamma)$ is comprised by a finite number of intervals (indeed, 1 or 2 intervals)

External Fields created by a couple of attractors

Main question: Which configurations of point masses ("charges" and "heights") are able to split the support into two intervals???

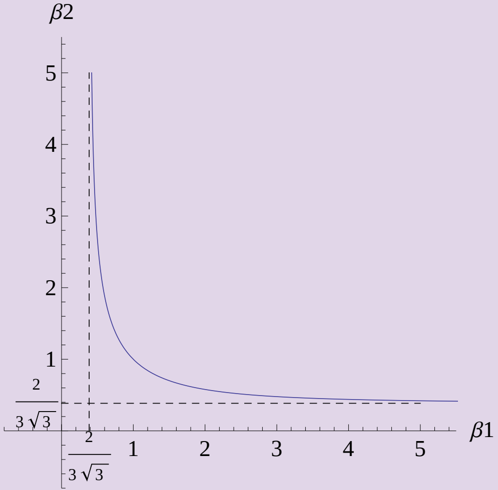
Look at the heights: β_1, β_2
Consider the bivariate polynomial:

$$f(x,y) = 27xy(x-y)^2 - 4(x^3+y^3) + 204xy(x+y) \\ - 48(x^2 - 7xy + y^2 + 4x + 4y) - 256$$

Critical curve in the (β_1, β_2)–plane: $f(\beta_1^2, \beta_2^2) = 0$

External Fields created by a couple of attractors

Critical curve



External Fields created by a couple of attractors

Main Result

- If $(\beta_1^2, \beta_2^2) \in \overline{\Omega_\infty} \implies S_\mu$ consists of a single interval ("one-cut") for any (λ_1, λ_2).
- If $(\beta_1^2, \beta_2^2) \in \Omega_0 \implies$ There exists a region in the (λ_1, λ_2)–plane for which S_μ consists of two disjoints intervals ("two-cut")

Remarks

- If the couple of masses is "quite far" from the real axis, then we have necessarily one-cut whatever the heights!
- The critical curve has horizontal and vertical asymptotes $\Longrightarrow$ If one of the masses is close enough to the real axis, then it is always possible to choose the charges to have two-cut.
- What finally determines the possible existence of a two-cut phase is the "degree of non-convexity" of the external field.

THANK YOU VERY MUCH!!!

See you...In Tenerife (why not?)!!!