We have the following relation
\[
\frac{\partial \psi(t_1, t_2)}{\partial m_j} = \left( \frac{2 \delta q(t_1, t_2)}{\partial m_j} - \frac{p_j}{\partial q(t_1, t_2)} \right) \frac{d t}{\partial q(t_1, t_2)}
\]
Therefore we have
\[
S(t_1, t_2) = \left( \frac{2 \delta q(t_1, t_2)}{\partial m_j} \right) \frac{d t}{\partial q(t_1, t_2)} dt.
\]

**Painlevé-I equation**

Painlevé-I equation is given by
\[
d_{\tau}q = \frac{d^2 \tau}{d t^2} \left( 2 \tau^2 - \tau \right).
\]
It is equivalent to the Hamiltonian system with Hamiltonian
\[
H = \frac{\tau^2}{2} - 2\tau q + a.
\]
We have the following relation
\[
\ln \tau(t_1, t_2) = S(t_1, t_2) + \left( \frac{2 \delta q}{\partial t} \right) \left( \frac{d t}{\partial q} \right) dt.
\]

**Painlevé-II equation**

Painlevé-II equation is given by
\[
d_{\tau}q = \frac{d^2 \tau}{d t^2} \left( 2 \tau^2 - \tau ^3 \right) + \frac{2}{t} (aq^2 + \beta + t).
\]
It is equivalent to the Hamiltonian system with Hamiltonian
\[
H = \frac{\tau^2}{4} - t^2 q^2 - 4aq + a.
\]
We have the following relation
\[
\ln \tau(t_1, t_2) = S(t_1, t_2) - \left( \frac{2 \delta q}{\partial t} \right) \left( \frac{d t}{\partial q} \right) dt.
\]

**Painlevé-III equation**

Painlevé-III equation is given by
\[
d_{\tau}q = \frac{d^2 \tau}{d t^2} \left( \frac{1}{2} \left( aq^2 + \beta + t \right) \right) + \gamma q^3 + \frac{\beta}{q}.
\]
where
\[
a = 8 \theta_0, \quad b = 4 - 8 \theta_0, \quad \gamma = 4, \quad \delta = -4.
\]
It is equivalent to the Hamiltonian system with Hamiltonian
\[
H = 2q^2 + 2t(q - 4\theta_0 + 4\theta_1) + \theta_1 \tau + \theta_0 \tau.
\]
We have the following relation
\[
\ln \tau(t_1, t_2) = S(t_1, t_2) + \left( \frac{1}{2} \frac{\delta \tau}{\delta q} \right) \left( \frac{\delta q}{\delta t} \right) dt + \left( \frac{1}{4} \frac{\delta \tau}{\delta \theta_0} \right) \left( \frac{\delta \theta_0}{\delta t} \right) dt.
\]

**Reference**
