Local geodesics for plurisubharmonic functions

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Motivations

1. **General goal:** 'good' transformations $u_0 \mapsto u_1$ of psh functions

2. **Global setting:** *metrics on Kähler manifolds* (X, ω)

$\omega > 0$ Kähler form
another Kähler form $\omega' = \omega + dd^c\varphi \in [\omega]$, so $\omega' \leftrightarrow \varphi$: *metrics*

Geodesics on the space of metrics: φ_t that minimize energy functional

$$\int_0^1 \int_X \dot{\varphi}_t^2 (\omega + dd^c \varphi_t)^n \, dt$$

(Mabuchi 1987, Semmes 1992, Donaldson 1997, Chen 2000...)

Characterization: φ_t is a geodesic $\Leftrightarrow (\omega + dd^c\Phi)^{n+1} = 0$ on $X \times S$
($n = \dim X$, $\Phi(z, \zeta) = \varphi_{\log|\zeta|}(z)$, and S is an annulus in $\mathbb{C}$)

Moreover, geodesics φ_t linearize *Mabuchi functional* $t \mapsto \mathcal{M}(\varphi_t)$.

A curve ψ_t is *subgeodesic* if the corresponding function Ψ satisfies
$(\omega + dd^c\Phi)^{n+1} \geq 0$. Mabuchi functional is convex on subgeodesics.

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Motivations: cont'd

3. **Further developments:** other functionals, singular metrics, ... (Berman, Berndtsson, Darvas, Guedj, Phong, Tian, Ross, Wytt Nyström...)

4. **Our aim:** local counterpart of the theory for functions on open sets.

Especially: applications?

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Especially: applications?

PSH

PSH(M): functions $u : M \to [-\infty, \infty)$ *plurisubharmonic* on a complex manifold M, i.e.:
(i) upper semicontinuous on M
(ii) $u \circ \phi$ subharmonic in the unit disk $\mathbb{D}$ for every holomorphic mapping $\phi : \mathbb{D} \to M$.

Basic examples:

1. $u = c \log |f|$ for any $c > 0$ and any holomorphic mapping $f : M \to \mathbb{C}^n$;
2. $u = \psi(\log |z_1|, \ldots, \log |z_n|)$ for a convex function ψ in $S \subset \mathbb{R}^n$.

Basic properties:

1. $u_k \in \mathrm{PSH}(M),\ 1 \le k \le N \Rightarrow u = \max_k u_k \in \mathrm{PSH}(M)$;
2. $u_k \in \mathrm{PSH}(M),\ u_k \searrow u \Rightarrow u \in \mathrm{PSH}(M)$;
3. $u_\alpha \in \mathrm{PSH}(M),\ u_\alpha < C\ \forall \alpha \Rightarrow u = \sup_\alpha^* u_\alpha \in \mathrm{PSH}(M)$.

Energy functional on Cegrell classes

$M = D \subset \mathbb{C}^n$: bounded hyperconvex domain.
Cegrell's class $\mathcal{E}_0(D)$: bounded plurisubharmonic functions u in D, $u|_{\partial D} = 0$ with finite total Monge-Ampère mass $\int_D (dd^c u)^n < \infty$.

Energy functional on $\mathcal{E}_0$:

$$\mathbf{E}(u) = \int_D u(dd^c u)^n.$$

Identity:

$$\mathbf{E}(u) - \mathbf{E}(v) = \int_D (u - v) \sum_{k=0}^{n} (dd^c u)^k \wedge (dd^c v)^{n-k}.$$

Corollary: If $u, v \in \mathcal{E}_0$ satisfy $u \leq v$, then $\mathbf{E}(u) \leq \mathbf{E}(v)$. If, in addition, $\mathbf{E}(u) = \mathbf{E}(v)$, then $u = v$ on D.

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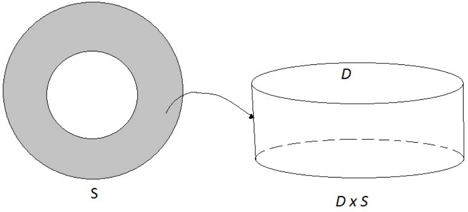
Geodesics for $\mathcal{E}_0$

$S = \{0 < \log|\zeta| < 1\} \subset \mathbb{C}, \quad S_j = \{\log|\zeta| = j\}, \quad \log|S| = (0,1)$



Given $u_0, u_1 \in \mathcal{E}_0(D)$, denote

$$W(u_0,u_1) = \{u \in PSH^-(D \times S) : \limsup_{\zeta \to S_j} u(\cdot,\zeta) \le u_j(\cdot),\ j = 0,1\}.$$

Definition. v_t is a *subgeodesic for* u_0, u_1 if $v_{\log|\zeta|} \in W(u_0,u_1)$.
The largest subgeodesic, u_t, is called *geodesic*: $u_t(z) = \widehat{u}(u, e^t)$, where $\widehat{u} = \sup\{u \in W(u_1,u_2)\} \in PSH^-(D \times S)$.

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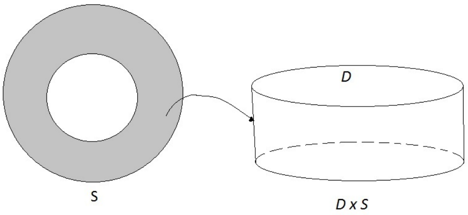

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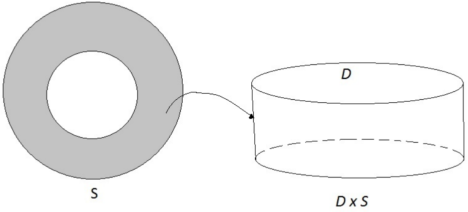

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Properties of geodesics:

1. $u_t \in \mathcal{E}_0(D)$
2. $u_t \leq (1-t)u_0 + tu_1$
3. $u_t \geq \max\{u_0 - M_1\, t, u_1 - M_0\,(1-t)\}$, where $M_j = \|u_j\|_\infty$.
4. $u_t \rightrightarrows u_j$ as $t \to j \in \{0,1\}$

Theorem

1. *The energy functional $u \mapsto \mathbf{E}(u) = \int_D u(dd^c u)^n$ is concave on $\mathcal{E}_0$.*
2. *For any subgeodesic v_t, $\mathbf{E}(v_t)$ is a convex function of t.*
3. *$t \mapsto \mathbf{E}(u_t)$ is linear iff u_t is a geodesic.*

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3. $t \mapsto \mathbf{E}(u_t)$ *is linear iff* u_t *is a geodesic.*

Sketch of the proof

Denote $\widehat{v}(z,\zeta) = v_{\log|\zeta|}(z)$.

Convexity of $\mathbf{E}(v_t)$ is equivalent to subharmonicity of the function

$$\widehat{\mathbf{E}} = \mathbf{E}(\widehat{v}) = \int_D \widehat{v}(d_z d_z^c \widehat{v})^n,$$

and the linearity of $\mathbf{E}$ corresponds to the harmonicity of $\widehat{\mathbf{E}}$.

All justifications hidden,

$$d_\zeta^c \widehat{\mathbf{E}} = (n+1)\int_D d_\zeta^c \widehat{v} \wedge (d_z d_z^c \widehat{v})^n$$

and

$$\begin{aligned} \frac{1}{n+1} d_\zeta d_\zeta^c \widehat{\mathbf{E}} &= \int_D d_\zeta d_\zeta^c \widehat{v} \wedge (d_z d_z^c \widehat{v})^n - n \int_D d_z d_\zeta^c \widehat{v} \wedge d_z^c d_\zeta \widehat{v} \wedge (d_z d_z^c \widehat{v})^{n-1} \\ &= \frac{1}{n+1} \int_D (dd^c \widehat{v})^{n+1}. \end{aligned}$$

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Uniqueness theorem

Corollary

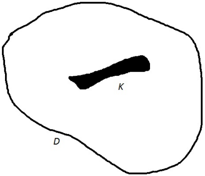
If $u_0, u_1 \in \mathcal{E}_0(D)$ *satisfy*

$$\int_D u_0 (dd^c u_0)^k \wedge (dd^c u_1)^{n-k} = \mathbf{E}(u_1), \quad k = 0, \dots, n,$$

then $u_0 = u_1$ *in* D.

Example: relative extremal functions

Let $K \Subset D$



Relative extremal function $\omega_K = \sup^*\{u \in \mathrm{PSH}^-(D) : \ u|_K \leq -1\}$.

We have: $\omega_K \in \mathcal{E}_0(D)$, $\mathbf{E}(\omega_K) = -\int_D (dd^c\omega_K)^n = -\mathrm{Cap}\,(K)$.

Let $u_j = u_{K_j}$, $j = 0, 1$. Then $\mathbf{E}(u_t) = (t-1)\,\mathrm{Cap}\,(K_0) - t\,\mathrm{Cap}\,(K_1)$.

Question: What is the geodesic u_t? Is $u_t = \omega_{K_t}$ for some K_t?

Answer: No.

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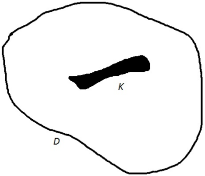

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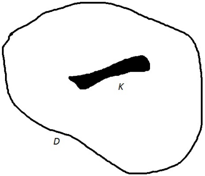

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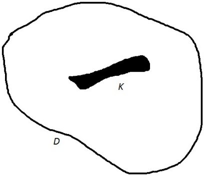

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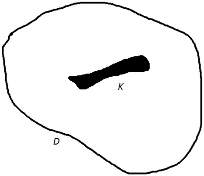

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1) $y \in D$ provided $z \in D$ and $|y_l| \leq |z_l|$ for all l,
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Let K_j also be compact Reinhardt subsets of D.
Then ω_{K_j} are toric (multi-circled) and so, the function $u_t(z)$ is convex in $(\log|z_1|, \ldots, \log|z_n|, t)$.

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$$K_t = K_0^{1-t} K_1^t = \{z : \ |z_l| = |\eta_l|^{1-t} |\xi_l|^t, \ 1 \le l \le n, \ \eta \in K_0, \ \xi \in K_1\}.$$

REF in toric setting

Let D be a bounded complete logarithmically convex Reinhardt domain:
1) $y \in D$ provided $z \in D$ and $|y_l| \leq |z_l|$ for all l,
2) $\log D = \{s \in \mathbb{R}^n_- : e^{s_1}, \ldots, e^{s_n} \in D\}$ is a convex subset of $\mathbb{R}^n$.

Let K_j also be compact Reinhardt subsets of D.
Then ω_{K_j} are toric (multi-circled) and so, the function $u_t(z)$ is convex in $(\log|z_1|, \ldots, \log|z_n|, t)$.

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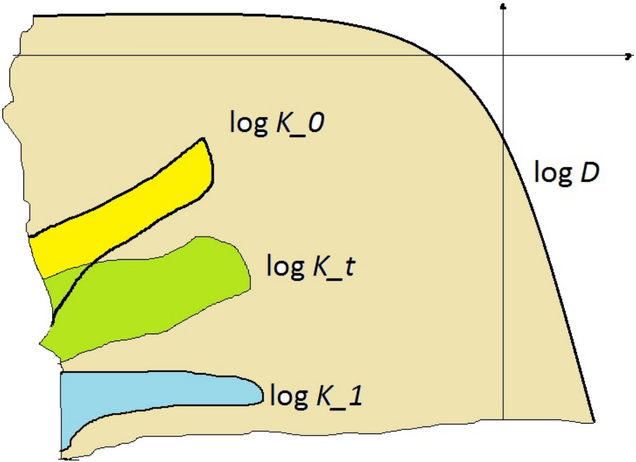

In other words, $\log K_t = (1-t)\log K_0 + t\log K_1$.

Brunn-Minkowski inequality

Recall: volumes $|\cdot|$ of convex combinations of two bodies $P_j \subset \mathbb{R}^n$ satisfy

$$|(1-t)P_0 + t\,P_1| \geq |P_0|^{1-t}\,|P_1|^t,$$

the Brunn-Minkowski inequality (in multiplicative form).

In our case, the sets $\log K_j$ typically are of infinite volume. Instead of the volumes, we have a *reversed Brunn-Minkowski inequality for the capacities* of K_t (multiplicative combinations of K_j), in additive form:

Theorem

In the toric case, the capacities of the sets K_t satisfy

$$\mathrm{Cap}\,(K_t) \leq (1-t)\,\mathrm{Cap}\,(K_0) + t\,\mathrm{Cap}\,(K_1).$$

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Example: geodesic of REFs is not REF

Let $n = 1$, $D = \mathbb{D}$, $K_0 = \{z : |z| \le e^{-1}\}$, $K_j = \{z : |z| \le e^{-2}\}$.
Then $K_t = \{z : |z| \le e^{-1-t}\}$.

The function

$$\omega_{K_t}(z) = \max\left\{\frac{\log|z|}{1+t}, -1\right\}$$

is not convex in $(\log|z|, t)$, so ω_{K_t} is not geodesic.

$\mathbf{E}(\omega_{K_t}) = -\mathrm{Cap}\,(K_t) = -(1+t)^{-1}$ is far from being linear.

Actually,

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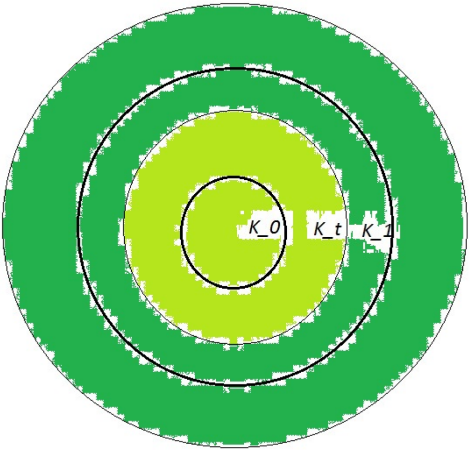

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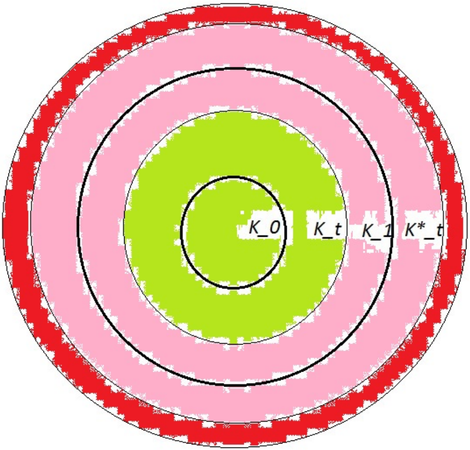

$$K_t = \{z : |z| \le e^{-1-t}\}, \quad K_t^* = \{z : |z| \le e^{t-1}\}$$

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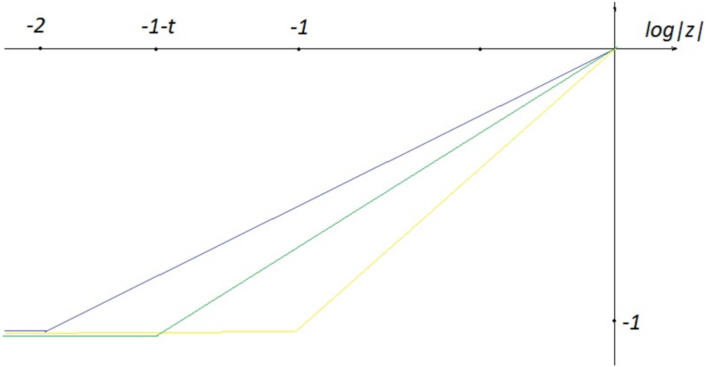

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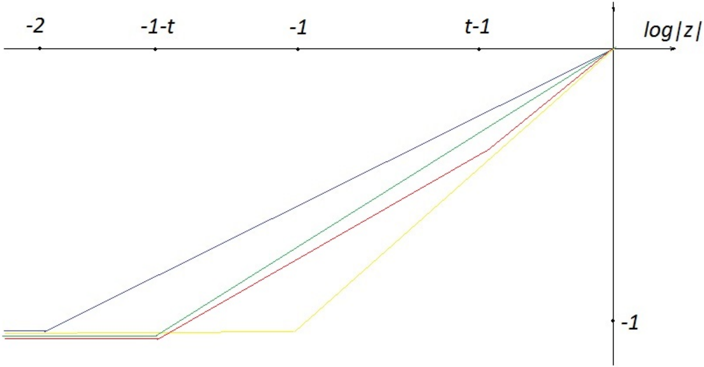

Example: geodesic of REFs is not REF, cont'd

In this example, the geodesic u_t still pertains some 'extremality' property: it is the *extremal function for the multi-plate condenser* (due to Poletsky) with the plates $K_t \subset K_t^* \subset K = \overline{\mathbb{D}}$.
Namely, u_t does not exceed preassigned constants on the plates and is a maximal psh function between the plates.

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Singular case

What happens if u_j are not bounded and can have singularities?

The construction can be extended to other classes of psh functions. If we still want to use the energy functional, we have to stick with Cegrell's energy classes - and then the whole picture (existence, linearity, uniqueness property) remains nearly the same.
The only new feature is that the end points u_j of the geodesic are attained in a weaker sense (convergence in capacity).

In larger classes, strong singularity can occur. The main message here is that essentially different singularities cannot be connected with (sub)geodesics.

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$\mathcal{F}_1$

$D \subset \mathbb{C}^n$: bounded hyperconvex domain.

Cegrell's class $\mathcal{F}_1(D)$: $u \in \mathrm{PSH}(D)$ that are limits of decreasing sequences $u_N \in \mathcal{E}_0(D)$ such that

$$\sup_N \int_D |u_N|(dd^c u_N)^n < \infty$$

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For $u \in \mathcal{F}_1(D)$,

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$\mathcal{F}_1$ (cont'd)

Like for $\mathcal{E}_0$, we still have the identity

$$\mathbf{E}(u) - \mathbf{E}(v) = \int_D (u - v) \sum_{k=0}^{n} (dd^c u)^k \wedge (dd^c v)^{n-k}$$

for $u, v \in \mathcal{F}_1(D)$ and the properties

$$u \leq v \quad \Leftrightarrow \quad \mathbf{E}(u) \leq \mathbf{E}(v),$$

and

$$\{u \leq v\} \ \& \ \{\mathbf{E}(u) = \mathbf{E}(v)\} \quad \Leftrightarrow \quad u = v.$$

Geodesics on $\mathcal{F}_1$

And the main result is valid on $u_j \in \mathcal{F}_1(D)$ as well:

Theorem

For any pair $u_0, u_1 \in \mathcal{F}_1(D)$ there exists a geodesic $u_t \subset \mathcal{F}_1(D)$, $0 < t < 1$, such that u_t converge in capacity to u_j as t approaches $j = 0$ and $j = 1$.
The energy functional $v \mapsto \mathbf{E}(v)$ is concave on $\mathcal{F}_1(D)$, while the function $t \mapsto \mathbf{E}(u_t)$ is linear on geodesics u_t and convex on subgeodesics $v_t \in \mathcal{F}_1(D)$.

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Any $u \in \mathrm{PSH}^-(D)$ is the limit of a decreasing sequence $u_N \in \mathcal{E}_0(D)$.

Let $u_j \in \mathcal{F}_1(D)$, $j = 0, 1$, and let $u_{j,N} \in \mathcal{E}_0(D)$ decrease to u_j as $N \to \infty$.

Then their geodesics $u_{t,N} \in \mathcal{E}_0(D)$ decrease to some functions v_t such that $v_{\log|\zeta|}(z) \in \mathrm{PSH}^-(D \times S)$.

Question: *How are v_t related to u_j?*

Easy to see: $\limsup_{t \to j} v_t \le u_j$.

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Relations to the Kähler case

Let (X, ω) be a compact Kähler manifold. An upper semicontinuous function φ on X is called *ω-plurisubharmonic* if $\omega + dd^c\varphi \geq 0$. Cegrell's classes were generalized to such functions by Guedj and Zeriahi. A corresponding class $\mathcal{E}_1(X, \omega)$ was introduced, and it has turned to be a natural frame for studying the Mabuchi functional (Berman, Boucksom, Guedj; Zeriahi).

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