Zeros of Asymptotically Extremal Polynomials

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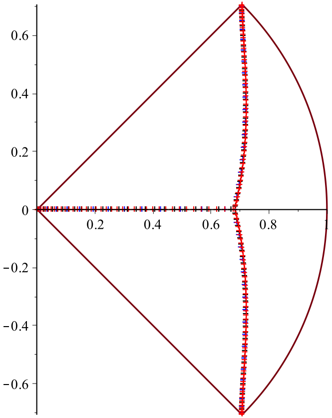
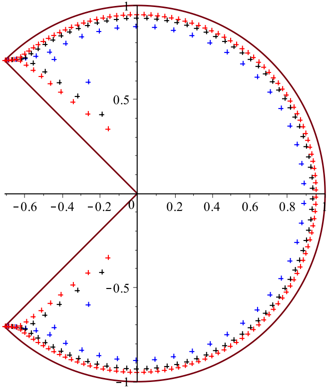


Zeros of Bergman polys for n=50, 100, 150
Sector opening $= \pi/2$



Zeros of Bergman polys for n=50, 100, 150
Sector opening $= 3\pi/2$

Definition

Let G be a bounded simply connected domain in the complex plane. A point z_0 on the boundary of G is said to be a **non-convex type singularity** (NCS) if it satisfies the following two conditions:

(i) There exists a closed disk $\overline{D}$ with z_0 on its circumference, such that $\overline{D}$ is contained in G except for the point z_0.

(ii) There exists a line segment L connecting a point ζ_0 in the interior of $\overline{D}$ to z_0 such that

$$\lim_{\substack{z \to z_0 \\ z \in L}} \frac{g_G(z, \zeta_0)}{|z - z_0|} = +\infty, \tag{1}$$

where $g_G(z, \zeta_0)$ denotes the Green function of G with pole at $\zeta_0 \in G$.

Theorem

Let $E \subset \mathbb{C}$ be a compact set of positive capacity, Ω the unbounded component of $\overline{\mathbb{C}} \setminus E$, and $\mathcal{E} := \overline{\mathbb{C}} \setminus \Omega$ denote the polynomial convex hull of E. Assume there is closed set $E_0 \subset \mathcal{E}$ with the following three properties:

(i) $\mathrm{cap}(E_0) > 0$;

(ii) *either* $E_0 = \mathcal{E}$ *or* $\mathrm{dist}(E_0, \mathcal{E} \setminus E_0) > 0$;

(iii) *either the interior* $\mathrm{int}(E_0)$ *of* E_0 *is empty or the boundary of each open component of* $\mathrm{int}(E_0)$ *contains an NCS point.*

Let V be an open set containing E_0 such that $\mathrm{dist}(V, \mathcal{E} \setminus E_0) > 0$ if $E_0 \neq \mathcal{E}$. Then for any asymptotically extremal sequence of monic polynomials $\{P_n\}_{n \in \mathcal{N}}$ for E,

$$\nu_{P_n}|_V \xrightarrow{\star} \mu_E|_{E_0}, \quad n \to \infty, \quad n \in \mathcal{N}, \tag{2}$$

where $\mu|_K$ denotes the restriction of a measure μ to the set K.

Definition

A measure μ is said to be an **electrostatic skeleton** for a compact E with $\mathrm{cap}(E) > 0$, if $\mathrm{supp}(\mu)$ has empty interior, connected complement, and $\mu^b = \mu_E$.

Conjecture

Every convex polygonal region has an electrostatic skeleton.