Orthogonal Polynomials on Polynomial Lemniscates

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Orthogonal Polynomials

- Let μ be a finite measure with compact and infinite support in $\mathbb{C}$.
- By performing Gram-Schmidt orthogonalization to $\{1, z, z^2, z^3, \ldots\}$, we arrive at the sequence of orthonormal polynomials $\{p_n(z;\mu)\}_{n\geq 0}$ satisfying

$$\int_{\mathbb{C}} p_n(z;\mu)\overline{p_m(z;\mu)}\mathrm{d}\mu(z) = \delta_{nm}.$$

- The leading coefficient of p_n is $\kappa_n = \kappa_n(\mu)$ and satisfies $\kappa_n > 0$.

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- $P_n(\cdot;\mu)$ satisfies

  $$\|P_n(\cdot;\mu)\|_{L^2(\mu)} = \inf\{\|Q\|_{L^2(\mu)} : Q = z^n + \text{ lower order terms}\},$$

  a property we call the *extremal property*.

The Bergman Shift

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- If we use the orthonormal polynomials as a basis for $\mathcal{P}$, then the matrix form of M_z is Hessenberg matrix:

$$M_z = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} & \cdots \\ M_{21} & M_{22} & M_{23} & M_{24} & \cdots \\ 0 & M_{32} & M_{33} & M_{34} & \cdots \\ 0 & 0 & M_{43} & M_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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- In the context of OPRL and OPUC, this is equivalent to studying properties of the recursion coefficients as $n \to \infty$.
- A common theme in both OPRL and OPUC is studying stability of the orthonormal polynomials under certain perturbations of the underlying measure.

A Simple Example

- If μ is arc-length measure on the unit circle then the Bergman Shift matrix is just the right shift operator on $\ell^2(\mathbb{N})$ and

$$\frac{p_n(z;\mu)}{p_{n+1}(z;\mu)} = \frac{1}{z}, \qquad |z| > 0, \quad n \geq 0$$

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- If μ satisfies $\mu'(\theta) > 0$ almost everywhere, then the Bergman Shift matrix converges along its diagonals to the right shift operator, and

$$\lim_{n\to\infty} \frac{p_n(z;\mu)}{p_{n+1}(z;\mu)} = \frac{1}{z}, \qquad |z| > 1.$$

Polynomial Lemniscates

- We will focus on the situation when the measure μ is concentrated near a set of the form

$$G_r := \{z \in \mathbb{C} : |Q(z)| \leq r\}$$

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  for some monic degree m polynomial Q and a positive real number r chosen so that each connected component of this set has smooth boundary.

- This is a natural generalization of OPUC, because the Green's function is $-\frac{1}{m}\log|Q(z)|$.

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- If μ is the equilibrium measure for G_r, then this set is already orthogonal, so the matrix $M_{Q(z)}$ with respect to this basis (for some subspace) is just a multiple of the right shift operator $\mathcal{R}$.

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- If we fill in this basis with good polynomial approximations to $\{\sqrt[m]{Q(z)^n}\}_{n\geq 1}$ and orthogonalize, then we expect the resulting matrix $M_{Q(z)} = Q(M_z)$ to be very close to a multiple of $\mathcal{R}^m$.

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- In some sense we can understand a general measure μ on G_r as a perturbation of the equilibrium measure by observing similarities of $Q(M_z)$ and a multiple of a power of $\mathcal{R}$.

Isospectral Torus

- If $\mathrm{supp}(\mu) \subseteq \mathbb{R}$

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & \cdots \\ 0 & 0 & a_3 & b_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- In the context of OPRL, one can easily identify the essential spectrum of the matrix J if the diagonals of J are q-periodic.

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- The map from q-periodic sequences to the polynomial Δ is far from injective. The preimage of a particular discriminant is known as the *isospectral torus* of e.
- A right limit of the matrix J is a doubly infinite matrix J_0 such that the sequence $\mathcal{L}^n J \mathcal{R}^n$ converges to J_0 pointwise as $n \to \infty$ through some subsequence.
- We say that J converges to the isospectral torus of e precisely when every right limit of J is in the isospectral torus of e.

Convergence to the Isospectral Torus (continued)

Magic Formula (Damanik, Killip, & Simon, 2010)

Let J_0 be a two-sided q-periodic Jacobi matrix with discriminant Δ_0 and essential spectrum e_0. If J_1 is another two-sided Jacobi matrix, then J_1 is in the isospectral torus of e_0 if and only if $\Delta_0(J_1) = \mathcal{L}^q + \mathcal{R}^q$.

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Theorem (Last & Simon, 2006)

If J converges to the isospectral torus for e_0, then the essential support of the spectral measure for J is e_0.

Convergence to the Isospectral Torus (continued)

Corollary

Suppose Δ_0 is the discriminant of a q-periodic Jacobi matrix and $e_0 = \Delta_0^{-1}([-2, 2])$. If

$$\lim_{n \to \infty} (\Delta_0(\mathcal{L}^n J \mathcal{R}^n))_{j,k} = (\mathcal{L}^q + \mathcal{R}^q)_{j,k}, \qquad j, k \in \mathbb{Z},$$

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then the essential support of the spectral measure for J is e_0.

- If the matrix J satisfies a certain asymptotic polynomial condition, then we deduce a similarity between the measure μ and the equilibrium measure for $\{x : |\mathrm{Re}[\Delta_0(x)]| \leq 2\}$.

Weak Asymptotic Measures

- The measures $\{|p_n(z;\mu)|^2 \mathrm{d}\mu(z)\}_{n\in\mathbb{N}}$ are all probability measures with support in a fixed compact set. Any weak limit is called a *weak asymptotic measure*.

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- The weak asymptotic measures reflect how effectively the orthonormal polynomials are able to "smooth out" the measure μ.

- The support of a weak asymptotic measure is concentrated near that portion of the measure that the orthonormal polynomials are least able to suppress.

Analog for General Measures

- An analog of the corollary exists for general measures.

Theorem (S., to appear in Constr. Approx.)

Let $Q(z)$ be a monic polynomial of degree m and let $\mathcal{G}$ be a banded Toeplitz matrix of width m. Suppose that the operators $\{(Q(M_z)-\mathcal{G})\mathcal{R}^n\}_{n\in\mathbb{N}}$ converge strongly to zero as $n\rightarrow\infty$ and

$$\lim_{n\rightarrow\infty}\left(\|\mathcal{G}^n e_{(n+3)m}\|\right)^{1/n}=r.$$

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$$\lim_{n\to\infty} \left(\|\mathcal{G}^n e_{(n+3)m}\|\right)^{1/n} = r.$$

Then every weak asymptotic measure γ is supported on $\{z : |Q(z)| \leq r\}$ and $\operatorname{supp}(\gamma) \cap \{z : |Q(z)| = r\} \neq \emptyset$.

Proof

- The strong convergence result easily implies $\|(Q(M_z)^k - \mathcal{G}^k)e_n\| \to 0$ as $n \to \infty$ for every $k \in \mathbb{N}$.

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- However

$$\lim_{n\to\infty} \|Q(M_z)^k e_n\|^2 = \lim_{n\to\infty} \int |Q(z)^k p_{n-1}(z;\mu)|^2 d\mu(z).$$

Proof (continued)

- Now take $n \to \infty$ through $\mathcal{N} \subseteq \mathbb{N}$ so the measures $|p_{n-1}(z;\mu)|^2 d\mu(z)$ converge weakly to γ.

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- If $\beta > r$ is such that $\gamma(\{z : |Q(z)| > \beta\}) = t > 0$, then we would have
$$\|\mathcal{G}^k e_{(k+3)m}\|^2 > \beta^{2k} t,$$
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- If $\beta < r$ is such that $\gamma(\{z : |Q(z)| \leq \beta\}) = 1$, then we would have
$$\|\mathcal{G}^k e_{(k+3)m}\|^2 \leq \beta^{2k},$$
which is a contradiction for large k.

Necessary and Sufficient Conditions

Theorem (S., to appear in Constr. Approx.)

Let μ be a finite measure with compact and infinite support and let Q be a polynomial of degree $m \geq 1$. Fix $r > 0$. The matrices $\{(Q(M_z) - r\mathcal{R}^m)\mathcal{R}^n\}_{n\in\mathbb{N}}$ converge strongly to 0 as $n \to \infty$ if and only if both of the following conditions are satisfied:

i) $\lim_{n\to\infty} \kappa_n \kappa_{n+m}^{-1} = r$,

ii) *every weak asymptotic measure is supported on* $\{z : |Q(z)| = r\}$.

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- The theorem applies to area measure on a polynomial lemniscate.

Summary

- We can study general measures in the complex plane from a perturbative viewpoint by examining the structure of the Bergman Shift matrix.
- In particular we can characterize those measures that are very heavily concentrated near the boundary of a polynomial lemniscate.
- For OPRL, a very nice result of this kind exists in the form of the *Magic Formula*.
- Our conclusion comes in the form of a statement about the supports of the weak asymptotic measures.