On the Variance of the Number of Roots of Complex Random Orthogonal Polynomials Spanned by OPUC†

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† The presented work is a preliminary report

Problem Statement

We study zeros of random orthogonal polynomials

$$P_n(z) = \sum_{j=0}^{n} \eta_j \varphi_j(z), \quad z \in \mathbb{C},$$

where n is a fixed integer, $\{\eta_j\}$ are complex-valued random variables, and $\{\varphi_j\}$ are orthogonal polynomials on the unit circle (OPUC).
As a reminder, the OPUC are polynomials $\{\varphi_j\}$ that are defined by a probability Borel measure μ on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} \varphi_n(e^{i\theta})\overline{\varphi_m(e^{i\theta})}\, d\mu(e^{i\theta}) = \delta_{nm}, \quad \text{for all } n, m \in \mathbb{N} \cup \{0\}. \tag{1}$$

For an annulus

$$A(s,t) = \{z \in \mathbb{C} : 0 \le s < |z| < t\},$$

the variance of the number of zeros of P_n in $A(s,t)$ is denoted as

$$\mathsf{Var}[N_n(A(s,t))] := \mathbb{E}[N_n(A(s,t))^2] - \mathbb{E}[N_n(A(s,t))]^2, \tag{2}$$

where $\mathbb{E}[N_n(A(s,t))]$ is the expected number of zeros of P_n in $A(s,t)$.
Problem (Due to Igor Pritsker): Find the following limit

$$\lim_{n\to\infty} \frac{\mathsf{Var}[N_n(A(s,t))]}{n}.$$

Types of Spanning OPUC Considered

The types of OPUC we consider as a spanning basis of P_n is directly linked to the recurrence relation for $\{\varphi_j\}$:

$$\varphi_{n+1}(z) = \frac{z\varphi_n(z) - \bar{\alpha}_n \varphi_n^*(z)}{\sqrt{1-|\alpha_n|^2}}, \quad n \in \mathbb{N} \cup \{0\},$$

where sequence of recurrence coefficients $\{\alpha_n\} \subset \mathbb{D}$, and $\varphi_n^*(z) = z^n \overline{\varphi_n(1/\bar{z})}$.
We consider the following classes of OPUC

1. $\{\varphi_j\}$ such that their associated recurrence coefficients $\{\alpha_j\}$ possess the property that
$$\sum_{j=0}^{\infty} |\alpha_j| < \infty.$$
We denote this class of OPUC as $\mathcal{M}_{\mathsf{AS}}$.
2. $\{\varphi_j\}$ that satisfy the property that locally uniformly for $z \in \mathbb{D}$ we have
$$\lim_{n\to\infty} \frac{\varphi_n(z)}{\varphi_n^*(z)} = 0, \tag{3}$$
which is known as the Nevai class of OPUC (note that Theorem 1.7.4 of [3] gives that the above limit is equivalent to $\lim_{n\to\infty} \alpha_n = 0$). We denote this class of OPUC as $\mathcal{M}_{\mathsf{N}}$.
3. $\{\varphi_j\}$ such that their associated measure μ of orthogonality in (1) is regular in the sense of Ullman-Stahl-Totik, that is,
$$\varepsilon_n := \frac{1}{n}\log|\kappa_n| \to 0, \quad \text{as} \quad n \to \infty, \tag{4}$$
where κ_n is the leading coefficient of φ_n (note that equation (1.5.22) of [3] gives the representation $\kappa_n = \prod_{i=0}^{n-1}(1-|\alpha_i|^2)^{-1/2}$). We denote this class of OPUC as $\mathcal{M}_{\mathsf{UST}}$.

We note the following hierarchy:

$$\mathcal{M}_{\mathsf{AS}} \subset \mathcal{M}_{\mathsf{N}} \subset \mathcal{M}_{\mathsf{UST}}.$$

Variance of the Number of Zeros in Annuli that contain the Unit Circle

Let $P_n(z) = \sum_{k=0}^{n} \eta_{k,n}\varphi_k(z)$, where $\{\varphi_k\}$ are OPUC, and $\{\eta_{k,n}\}$ are complex-valued random variables such that

$$\sup\{\mathbb{E}[|\eta_{k,n}|^t] \mid k = 0,1,\ldots,n;\ n \in \mathbb{N}\} < \infty, \quad t \in (0,1] \tag{5}$$

and

$$\min\left(\inf_{n\in\mathbb{N}} \mathbb{E}[\log|\eta_{n,n}|],\ \inf_{n\in\mathbb{N}, z\in\mathbb{C}} \mathbb{E}[\log|\eta_{0,n} + z|]\right) > -\infty. \tag{6}$$

Relying on estimates concerning the expected discrepancy, $\mathbb{E}[|N_n(A(1/r,r))/n - 1|]$, of P_n provided in Theorem 3.1 of [2], with further estimation it follows that the variance of the number of zeros given by (2) satisfies the following:

Theorem

Taking $r \in (0,1)$, for $P_n(z) = \sum_{k=0}^{n} \eta_{k,n}\varphi_k(z)$ with $\{\eta_{k,n}\}$ satisfying (5) and (6), it follows that

1. *When $\{\varphi_j\} \subset \mathcal{M}_{\mathsf{AS}}$, we have*
$$\frac{\mathsf{Var}[N_n(A(1/r,r))]}{n^2} = \mathcal{O}\left(\sqrt{\frac{\log n}{n}}\right), \quad \text{as} \quad n \to \infty.$$
2. *When $\{\varphi_j\} \subset \mathcal{M}_{\mathsf{UST}}$, we have*
$$\frac{\mathsf{Var}[N_n(A(1/r,r))]}{n^2} = \mathcal{O}\left(\max\left\{\sqrt{\frac{\log n}{n}}, \varepsilon_n^{1/4}\right\}\right), \quad \text{as} \quad n \to \infty,$$
where ε_n is given by (4).

Lemmas concerning Annuli that do not contain the Unit Circle

Due to the linearity of the expectation, observe that (2) can be written as

$$\mathsf{Var}[N_n(A(s,t))] = \mathbb{E}[N_n(A(s,t))] + \mathbb{E}[N_n(A(s,t))(N_n(A(s,t))-1)] - \mathbb{E}[N_n(A(s,t))]^2. \tag{7}$$

When $P_n(z) = \sum_{k=0}^{n} \eta_k\varphi_k$, with $\{\eta_k\}$ i.i.d. complex-valued standard Gaussian, and $\{\varphi_k\}$ are OPUC, one can appeal to integral formulas for $\mathbb{E}[N_n(A(s,t))]$ and $\mathbb{E}[N_n(A(s,t))(N_n(A(s,t))-1)]$ given by Corollary 3.4.2 in [1].
Combining the formula for $\mathbb{E}[N_n(A(s,t))]$ with Corollary 2 of [4] gives the following:

Lemma

When $\{\varphi_j\} \subset \mathcal{M}_{\mathsf{N}}$ and $A(s,t)$ does not contain the unit circle, we have

$$\lim_{n\to\infty} \mathbb{E}[N_n(A(s,t))] = \frac{1}{\pi}\int_{A(s,t)} \frac{1}{(1-|z|^2)^2}\, dA(z). \tag{8}$$

Appealing to the formula for $\mathbb{E}[N_n(A(s,t))(N_n(A(s,t))-1)]$, after much algebraic simplification then using the limit (3), it follows that:

Lemma

For $\{\varphi_j\} \subset \mathcal{M}_{\mathsf{N}}$ and $A(s,t)$ not containing the unit circle, we have

$$\lim_{n\to\infty} \mathbb{E}[N_n(A(s,t))(N_n(A(s,t))-1)] = \frac{1}{\pi^2}\int_{A(s,t)}\int_{A(s,t)} \left(\frac{1}{(1-|z|^2)^2(1-|w|^2)^2} - \frac{1}{|1-z\overline{w}|^4}\right) dA(z)\, dA(w) \tag{9}$$

Variance of the Number of Zeros in Annuli that do not contain the Unit Circle

Using the representation (7) and combining (8) with (9), we achieve the following:

Theorem

Let $P_n(z) = \sum_{k=0}^{n} \eta_k\varphi_k(z)$, where $\{\varphi_k\} \subset \mathcal{M}_{\mathsf{N}}$, and $\{\eta_k\}$ are i.i.d. complex-valued standard Gaussian random variables. Then

$$\lim_{n\to\infty} \mathsf{Var}[N_n(A(s,t))] = \begin{cases} \dfrac{(t^2-s^2)[1-s^2(t^4(2+s^2)-2)]}{(1-t^4)(1-s^4)(1-(st)^2)}, & A(s,t) \subsetneq \mathbb{D}, \\[2ex] \dfrac{(t^2-s^2)[1-t^2(s^4(2+t^2)-2)]}{(1-t^4)(1-s^4)(1-(st)^2)}, & A(s,t) \subsetneq \mathbb{C}\setminus\overline{\mathbb{D}}. \end{cases}$$

We note that taking $s = 0$ and $t < 1$ in the above theorem, we achieve that the random orthogonal polynomial possesses the property that

$$\lim_{n\to\infty} \mathsf{Var}[N_n(D(0,t))] = \frac{t^2}{1-t^4}, \quad \text{where} \quad D(0,t) = \{z \in \mathbb{C} : |z| < t\}.$$

Conjectures/Work in Progress

1. Under suitable conditions on $\{\varphi_j\}$ and $\{\eta_j\}$, we have
$$\lim_{n\to\infty} \frac{\mathsf{Var}[N_n(\mathbb{D})]}{n} = c,$$
where c is a positive constant†.
2. Under suitable conditions on $\{\varphi_j\}$ and $\{\eta_j\}$, it follows that
$$\frac{N_n(\mathbb{D}) - \mathbb{E}[N_n(\mathbb{D})]}{\sqrt{\mathsf{Var}[N_n(\mathbb{D})]}} \xrightarrow{d} N(0,1), \quad \text{as} \quad n \to \infty.$$

† The statement of the conjecture is due to Igor Pritsker.

References

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