The dressing method and solutions to integrable systems

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The Korteweg-de Vries equation

The KdV equation on $u(x,t)$:

$$u_t = \frac{3}{2}uu_x + \frac{1}{4}u_{xxx}.$$

The KdV and related equations occur in many areas of mathematics:

- Physically, the KdV equation describes weakly nonlinear waves in various media, such as shallow water waves.
- KdV was the first equation in the modern theory of integrable systems.
- Counting problems in algebraic geometry.

Major open problem: For what classes of initial data can we solve the initial value problem for KdV?

Lax representation for KdV

The KdV equation has a Lax representation:

$$\frac{\partial L}{\partial t} = [L, A],$$

where L is the Schrödinger operator and A is an auxiliary operator

$$L = -\partial_x^2 + u, \quad A = \partial_x^3 - \frac{3}{2} u \partial_x - \frac{3}{4} u_x = [(-L)^{3/2}]_+.$$

KdV is the consistency condition for an overdetermined linear system:

$$L\psi = E\psi, \quad \partial_t \psi = A\psi,$$

on a complex-valued function $\psi(x, E, t)$, where E is a spectral parameter.

The time evolution preserves the spectrum of L, and the study of KdV is closely related to the spectral theory of L.

Spectral theory of L and the initial value problem for KdV

To solve the initial value problem for KdV, we need to study the spectral theory of the one-dimensional Schrödinger operator L:

$$L\psi = [-\partial_x^2 + u(x)]\psi = E\psi, \quad \psi \text{ bounded.}$$

There are two important classes of potentials $u(x)$ for which the spectral theory of L is well-understood, and the corresponding initial value problem has an effective solution:

If $u(x)$ vanishes sufficiently fast as $x \to \pm\infty$, we can solve the initial value problem for KdV by using the *inverse scattering transform* (IST).

If $u(x)$ is periodic, we can approximate it and solve the initial value problem by using *finite-gap potentials*.

Motivating question. What is the relationship between the IST and finite-gap solutions?

$u(x)$ rapidly vanishing: scattering data

Suppose that $u(x)$ rapidly vanishes at infinity:

$$u(x) = O(1/x^{2+\varepsilon}), \quad x \to \pm\infty.$$

We consider the Schrödinger equation

$$L\psi = [-\partial_x^2 + u(x)]\psi = E\psi, \quad \psi \text{ bounded on } \mathbb{R}.$$

For $E = k^2 \geq 0$, the solution space has dimension 2, so there is a solution

$$\psi(x,k) = \begin{cases} e^{-ikx} + c(k)e^{ikx} + o(1) & \text{as } x \to +\infty, \\ d(k)e^{-ikx} + o(1) & \text{as } x \to -\infty. \end{cases}$$

For finitely many negative $E = -\kappa_n^2$, $n = 1, \ldots, N$, there is one solution:

$$\psi_n(x) = \begin{cases} e^{\kappa_n x}(1 + o(1)) & \text{as } x \to -\infty, \\ e^{-\kappa_n x}(b_n + o(1)) & \text{as } x \to \infty. \end{cases}$$

The set $s = \{c(k), \kappa_n, b_n\}$ is the *scattering data* of the potential $u(x)$.

GGKM equations and the inverse scattering transform

If $u(x,t)$ satisfies KdV, then the spectral data $s(t)$ evolves trivially:

$$c(k,t) = c(k)e^{8ik^3 t}, \quad \kappa_n(t) = \kappa_n, \quad b_n(t) = b_n e^{8\kappa_n^3 t}.$$

We can solve the initial value problem for KdV for vanishing $u(x)$:

$$u(x,0) \to s(0) \to s(t) \to u(x,t).$$

We can reconstruct $u(x,t)$ from its scattering data $s = \{c(k), \kappa_n, b_n\}$ using the inverse scattering transform.

Introduce the function $F(x,t)$, where M_n is the L_2-norm $\psi_n(x)$.

$$F(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k,t)e^{ikx}dk + \sum_{n=1}^{N} M_n^2 e^{-\kappa_n x},$$

where the M_n are the L_2-norms of the eigenfunctions $\psi_n(x)$.

Solve the Marchenko equation for $K(x,y,t)$:

$$K(x,y,t) + F(x+y,t) + \int_x^{\infty} K(x,z,t)F(z+y,t)dz = 0.$$

Find the potential

$$u(x,t) = -\partial_x K(x,x,t).$$

Bargmann potentials and N-soliton solutions of KdV

The Marchenko equation can be solved explicitly when $c(k) = 0$.

If $s = \{0, \kappa_n, b_n\}$, $n = 1, \ldots, N$, then $u(x)$ is a *reflectionless Bargmann potential* and $u(x, t)$ is an *N-soliton solution* of KdV.

For $N = 1$ we get a traveling solitary wave:

$$-u(x,t) = \frac{2\kappa^2}{\cosh^2 \kappa(x - 4\kappa^2 t - x_0)}.$$

In general we have N interacting solitary waves, given by the Bargmann formula

$$-u(x,t) = 2\partial_x^2 \log \det |M_{nm}|,$$

$$M_{nm} = \delta_{nm} + c_n e^{8\kappa_n^3 t} \frac{e^{-(\kappa_n+\kappa_m)x}}{\kappa_n + \kappa_m}, \quad c_n = \frac{b_n}{ia'(i\kappa_n)} > 0, \quad a(k) = \prod_{n=1}^{N} \frac{k - i\kappa_n}{k + i\kappa_n}.$$

$u(x)$ periodic: finite-gap theory

Suppose that $u(x)$ is periodic:

$$u(x+T) = u(x).$$

We consider the Schrödinger equation

$$L\psi = [-\partial_x^2 + u(x)]\psi = E\psi, \quad \psi \text{ bounded on } S^1 = \mathbb{R}/T.$$

The spectrum of L is described by Bloch–Floquet theory consists of an infinite sequence of closed intervals

$$\mathcal{S} = [\lambda_1, \lambda_2] \cup [\lambda_3, \lambda_4] \cup [\lambda_5, \lambda_6] \cup \cdots, \quad \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

For each $E \in \mathcal{S}$, there is a two-dimensional space of solutions (one-dimensional at boundary points λ_i).

The eigenfunction $\psi(x, k)$ is defined on the *spectral curve* C: a hyperelliptic Riemann surface of infinite genus that is a double cover of the complex plane branched over the points $\lambda_1, \lambda_2, \ldots$

Finite-gap potentials

For an L^2-dense subset of periodic potentials, the spectrum has only finitely many gaps

$$S = [\lambda_1, \lambda_2] \cup \cdots \cup [\lambda_{2g-2}, \lambda_{2g-1}] \cup [\lambda_{2g}, \infty)$$

The spectral curve C is an algebraic Riemann surface of genus g.

The eigenfunction $\psi(x, k)$ has a pole divisor D of degree g on C.

$\psi(x, k)$ and $u(x)$ can be reconstructed from C and D.

If $u(x, t)$ satisfies KdV, then C does not depend on t, while D evolves linearly on the Jacobian variety $\mathrm{Jac}(C)$. The solution is given by the Matveev–Its formula

$$u(x, t) = 2\partial_x^2 \ln \theta(xU + tV + Z) + c,$$

where θ is the theta function of $\mathrm{Jac}(C)$.

For generic spectral data, this solution is quasi-periodic in x and t.

Genus one solutions

The solutions corresponding to genus one curves can be found by looking for traveling wave solutions of KdV:

$$\frac{1}{4}u_{xxx} = \frac{3}{2}uu_x - u_t, \quad u(x,t) = f(x-ct).$$

$$f''' = 6ff' + 4cf'$$

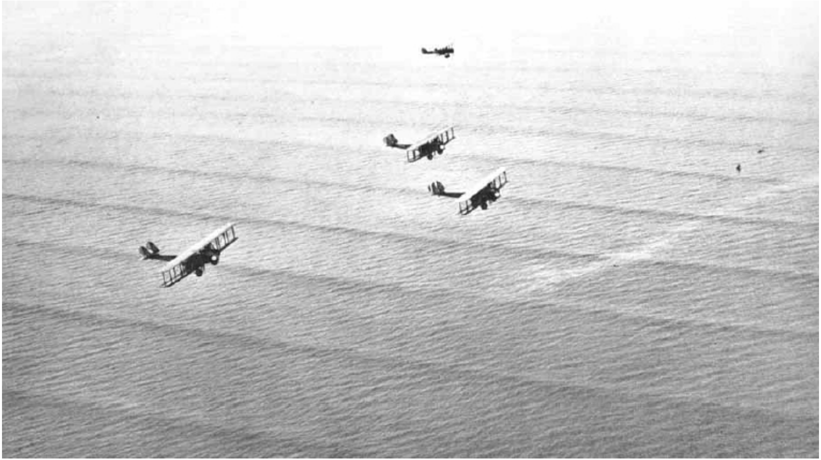
$$f'' = 3f^2 + 4cf + c_1,$$

$$\frac{1}{2}(f')^2 = f^3 + 2cf^2 + c_1 f + c_2.$$

We solve this in terms of the Weierstrass function $\wp$ of the associated elliptic curve and obtain the *cnoidal wave* solution, known since the 19th century:

$$u(x,t) = 2\wp(x + i\omega' - ct) + \text{const}$$

Cnoidal wave



$$u(x,t) = 2\wp(x + i\omega' - ct) + \text{const}$$

IST and finite-gap solutions

What is the relationship between the IST and finite-gap solutions?

Mumford: degenerating the spectral curve to a rational nodal curve reduces N-gap solutions to N-soliton solutions.

Idea. View finite-gap solutions as limits of soliton solutions as $N \to \infty$.

Lundina, Marchenko: Proved that periodic finite-gap solutions are contained in a suitable closure of the set of N-soliton solutions (no effective formulas).

Key difference. The finite-gap method is symmetric in $x \to -x$, while the IST is not. We can define an equivalent version of IST by considering the scattering from the left, but there is a choice to be made.

Previous work

Krichever: a partial degeneration gives solitons on a finite-gap background.

Egorova, Grunert, Teschl: inverse scattering transform on a finite-gap background.

Trogdon, Deconinck: Riemann–Hilbert problem for finite-gap solutions and finite-gap solutions plus solitons.

Binder, Damanik, Goldstein, Lukic: proved the existence of the solution of the initial value problem for a certain class of quasi-periodic initial data.

Motivation: Fourier transform vs. d'Alembert's formula

There are two approaches to the wave equation

$$u_{tt} = u_{xx}, \quad -\infty < x < \infty.$$

For initial data $u(x,0) = A(x)$, $u_x(x,0) = B(x)$, we find their Fourier transforms, apply time evolution, and then find the inverse Fourier transform.

Alternatively we can use the general formula

$$u(x,t) = f(x+t) + g(x+t),$$

which is local in x and t. Matching the initial data gives d'Alembert's formula:

$$u(x,t) = \frac{1}{2}[A(x-t) + A(x+t)] + \frac{1}{2}\int_{x-t}^{x+t} B(s)ds.$$

The IST is a nonlinear version of the Fourier transform.

The dressing method is as a nonlinear version of d'Alembert's formula.

The dressing method

The idea of the dressing method is to construct solutions $u(x)$ of KdV by specifying the analytic properties of the corresponding eigenfunction of the Schrödinger equation:

$$-\psi_{xx} + u(x)\psi = k^2\psi, \quad \psi(x,k) \to e^{-ikx} \text{ as } |k| \to \infty.$$

Substitute $\psi(x,k) = \chi(x,k)e^{-ikx}$:

$$\chi_{xx} - 2ik\chi_x - u(x)\chi = 0, \quad \chi(x,k) \to 1 \text{ as } |k| \to \infty.$$

We encode the analytic properties of χ in a $\overline{\partial}$-problem:

$$\frac{\partial \chi}{\partial \overline{k}} = ie^{2ikx}T(k)\chi(-k,x), \quad T(\overline{k}) = -\overline{T(-k)}.$$

The corresponding solution of KdV is equal to

$$u(x) = 2\frac{d}{dx}\chi_0(x), \quad \chi(x,k) = 1 + \frac{i\chi_0(x)}{k} + \cdots$$

Adding time dependence corresponds to replacing $2ikx$ with $2ikx + 8ik^3t$.

Analytic properties of χ

The class of initial data determines the analytic properties of χ:

If $u(x)$ is a Bargmann potential, then χ is rational with simple poles on the negative imaginary axis.

If $u(x)$ is rapidly vanishing, then χ has poles on the negative imaginary axis and a jump along the real axis.

If $u(x)$ is finite-gap, then χ has jumps along the imaginary axis and lifts to an algebraic function on the corresponding hyperelliptic curve.

Bargmann potentials via dressing method, 1st attempt

If $u(x)$ is a Bargmann potential with spectral data $s = \{0, \kappa_n, c_n\}$, then χ is a rational function with simple poles along the negative imaginary axis at $-i\kappa_n$:

$$\chi(x,k) = 1 + i\sum_{n=1}^{N} \frac{\chi_n(x)}{k - i\kappa_n}.$$

This function satisfies the $\overline{\partial}$-problem

$$\frac{\partial\chi}{\partial\bar{k}} = ie^{2ikx}T(k)\chi(-k,x), \quad T(k) = \sum_{n=1}^{N} c_n\delta(k - i\kappa_n).$$

The $\chi_n(x)$ and $u(x)$ are determined by the system

$$\chi_n(x) = c_n\chi(x, -i\kappa_n)e^{-2\kappa_n x}, \quad u(x) = 2\frac{d}{dx}\sum_{n=1}^{N}\chi_n(x)$$

Naive limit $N \to \infty$: replace poles with cuts

Krichever, 1980s: define the limit $N \to \infty$ by allowing the poles of χ to coalesce into a jump along the negative imaginary axis.

The function χ then satisfies a singular integral equation, and its approximations by Riemann sums produce N-soliton solutions.

The resulting potentials $u(x)$ are bounded as $x \to -\infty$ but are decreasing as $x \to +\infty$.

We drop the physical assumption that there are poles only along the negative part of the imaginary axis.

Bargmann potentials via dressing method, 2nd attempt

Let $\kappa_1, \ldots, \kappa_N$ and $c_1, \ldots, c_n$ be nonzero real numbers satisfying $\kappa_m \neq \pm\kappa_n$ for all $m \neq n$, $c_n/\kappa_n > 0$ for all n. Consider the $\overline{\partial}$-problem

$$\frac{\partial \chi}{\partial \overline{k}} = ie^{2ikx} T(k)\chi(-k, x), \quad T(k) = \sum_{n=1}^{N} c_n \delta(k - i\kappa_n).$$

There is a unique rational function χ satisfying this problem:

$$\chi(x, k) = 1 + i \sum_{n=1}^{N} \frac{\chi_n(x)}{k - i\kappa_n}, \quad \chi_n(x) = c_n \chi(x, -i\kappa_n) e^{-2\kappa_n x}.$$

The corresponding potential $u(x)$ is a reflectionless Bargmann potential with spectrum $\{-\kappa_1^2, \ldots, -\kappa_N^2\}$. Furthermore, for each n, replacing

$$\widetilde{\kappa}_i = \begin{cases} \kappa_i, & i \neq n, \\ -\kappa_n, & i = n, \end{cases} \quad \widetilde{c}_i = \begin{cases} \left(\dfrac{\kappa_i - \kappa_n}{\kappa_i + \kappa_n}\right)^2 c_i, & i \neq n, \\ -4\pi^2 \kappa_n^2 / c_n, & i = n, \end{cases}$$

does not change the potential $u(x)$.

The limit $N \to \infty$: replace poles with cuts

Fix $0 < k_1 < k_2$, and let R_1 and R_2 be two positive functions on $[k_1, k_2]$. Consider the kernel

$$T(k) = i\delta(k_R)[R_1(k_I) - R_2(-k_I)], \quad k = k_R + ik_I$$

We consider a function χ satisfying the $\overline{\partial}$-problem

$$\frac{\partial \chi}{\partial \overline{k}} = ie^{2ikx} T(k)\chi(-k, x).$$

It is analytic on the k-plane except for two cuts $[ia, ib]$ and $[-ib, -ia]$.

Equivalently, we are solving a RH problem on $\Xi(k) = [\chi(k) \ \ \chi(-k)]^T$:

$$\Xi^+(ip) = M(p)\Xi^-(ip), \quad \Xi^+(-ip) = M^T(p)\Xi^-(-ip), \quad p \in [a, b],$$

$$M(x, t, p) = \frac{1}{1 + R_1 R_2} \begin{bmatrix} 1 - R_1 R_2 & 2iR_1 e^{-2px - 8p^3 t} \\ 2iR_2 e^{2px + 8p^3 t} & 1 - R_1 R_2 \end{bmatrix}$$

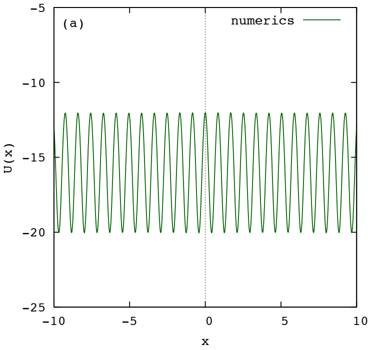
The corresponding solution $u(x, t)$ of the KdV equation

$$u(x, t) = 2\partial_x \chi_0(x, t), \quad \chi(x, t, k) = 1 + \frac{i\chi_0(x, t)}{k} + O(k^{-2})$$

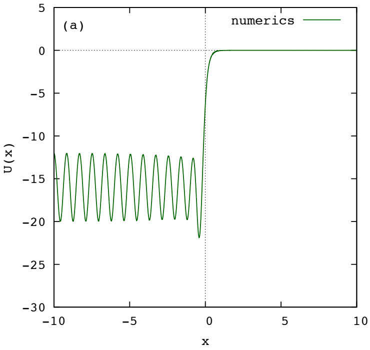
is bounded as $x \to \pm\infty$ and has the spectrum $[-b^2, -a^2] \cup [0, \infty)$.

Numerical simulations for constant R_1 and R_2

We can approximately solve the Riemann–Hilbert problem using N-soliton solutions. We only consider constant R_1 and R_2 on $[a, b] = [2, 4]$.



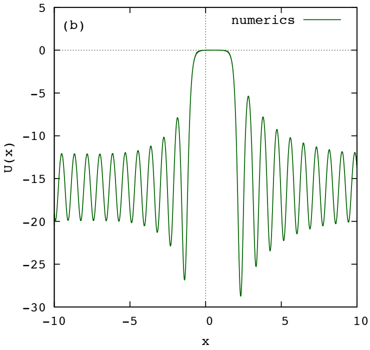
$R_1 = 1, \quad R_2 = 1$



$R_1 = 1, \quad R_2 = 0$

Numerical simulations for constant R_1 and R_2

We can approximately solve the Riemann–Hilbert problem using N-soliton solutions. We only consider constant R_1 and R_2 on $[a, b] = [2, 4]$.



$$R_1 = 10^{-3}, \quad R_2 = 10^{-6}$$

The Kadomtsev–Petviashvili equation

The KP-II equation describes quasi-one-dimensional shallow water waves:

$$\frac{\partial}{\partial x}\left(-4\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+6u\frac{\partial u}{\partial x}\right)+3\frac{\partial^2 u}{\partial y^2}=0.$$

It has the following Lax representation

$$[\partial_y - L, \partial_t - A] = 0,$$

where L and A are the same auxiliary operators as for KdV:

$$L = -\partial_x^2 + u, \quad A = \partial_x^3 - \frac{3}{2}u\partial_x - \frac{3}{4}u_x = [(-L)^{3/2}]_+.$$

Solutions of KP-II via the dressing method

We consider the following $\bar{\partial}$-problem on a function $\chi(k, r)$, where $r = (x, y, t)$:

$$\frac{\partial \chi(k, r)}{\partial \bar{k}} = \pi \delta(b) \int_{-\infty}^{\infty} \chi(\alpha, r) R_0(\alpha, a) e^{\Phi(\alpha, r) - \Phi(a, r)} d\alpha,$$

$$k = a + bi, \quad \Phi(k, r) = kx + k^2 y + k^3 t, \quad \overline{R_0(\alpha, a)} = R_0(\alpha, a).$$

The function χ has a jump along the real axis. If the $\bar{\partial}$-problem has a unique solution, then

$$u = 2 \frac{\partial \chi_1}{\partial x}, \quad \chi(k, r) = 1 + \frac{\chi_1(r)}{k} + O\left(\frac{1}{k^2}\right)$$

is a real-valued solution of the KP-II equation.

Degenerate dressing kernel

The function χ satisfies the following $\overline{\partial}$-problem:

$$\frac{\partial\chi(k,r)}{\partial\bar{k}} = \pi\delta(b)\int_{-\infty}^{\infty}\chi(\alpha,r)R_0(\alpha,a)e^{\Phi(\alpha,r)-\Phi(a,r)}d\alpha,$$

We consider a kernel of the following form:

$$R_0(\alpha,a) = \sum_{n=1}^{N} f_n(\alpha)g_n(a)$$

with linearly independent functions $g_n(a)$. Substituting

$$\chi(k,r) = 1 + \int_{-\infty}^{\infty}\frac{\varphi(a,r)e^{-\Phi(a,r)}}{k-s}ds, \quad \varphi(a,r) = \sum_{n=1}^{N}\varphi_n(r)g_n(a),$$

we obtain a linear system on the φ_n which we can solve explicitly, and obtain a solution of KP-II.

Solution with degenerate dressing kernel

The following function $u(x, y, t)$ satisfies the KP-II equation:

$$u(x,y,t) = 2\partial_x^2 \log \begin{vmatrix} 1+\partial_x^{-1}F_1G_1 & \partial_x^{-1}F_1G_2 & \cdots & \partial_x^{-1}F_1G_N \\ \partial_x^{-1}F_2G_1 & 1+\partial_x^{-1}F_2G_2 & \cdots & \partial_x^{-1}F_2G_N \\ \cdots & \cdots & \cdots & \cdots \\ \partial_x^{-1}F_NG_1 & \partial_x^{-1}F_NG_2 & \cdots & 1+\partial_x^{-1}F_NG_N \end{vmatrix}.$$

Here $F_n(r)$ and $G_n(r)$ are

$$F_n(r) = \int_{-\infty}^{\infty} f_n(\alpha)e^{\Phi(\alpha,r)}d\alpha, \quad G_n(r) = \int_{-\infty}^{\infty} g_n(a)e^{-\Phi(a,r)}da.$$

These functions satisfy:

$$\frac{\partial F_n}{\partial y} = \frac{\partial^2 F_n}{\partial x^2}, \quad \frac{\partial F_n}{\partial t} = \frac{\partial^3 F_n}{\partial x^3},$$

$$\frac{\partial G_n}{\partial y} = -\frac{\partial^2 G_n}{\partial x^2}, \quad \frac{\partial G_n}{\partial t} = \frac{\partial^3 G_n}{\partial x^3}.$$

The Wronskian method

There is a different method of constructing solutions of the KP-II equation (Freeman, Nimmo). Let $\widetilde{F}_1, \dots, \widetilde{F}_M$ be a linearly independent set of solutions of the system

$$\frac{\partial \widetilde{F}_n}{\partial y} = \frac{\partial^2 \widetilde{F}_n}{\partial x^2}, \quad \frac{\partial \widetilde{F}_n}{\partial t} = \frac{\partial^3 \widetilde{F}_n}{\partial x^3},$$

Then their Wronskian is a solution of KP-2:

$$u(r) = 2\partial_x^2 \log \mathrm{Wr}(\widetilde{F}_1, \dots, \widetilde{F}_M) = 2\partial_x^2 \log \begin{vmatrix} \widetilde{F}_1^{(0)} & \cdots & \widetilde{F}_M^{(0)} \\ \cdots & \cdots & \cdots \\ \widetilde{F}_1^{(M-1)} & \cdots & \widetilde{F}_M^{(M-1)} \end{vmatrix}.$$

We do not know what is the relationship between these methods.

Two examples with $N = 1$

We assume that R has finite support and that χ is a rational function.

Suppose that

$$f(\alpha) = \sum_{i=1}^{N_1} C_i \delta(\alpha - \alpha_i), \quad g(a) = \delta(a - a_0), \quad R(\alpha, a) = f(\alpha) g(a).$$

We get the following solution of KP-II:

$$u = 2\partial_x^2 \log \left[1 + \sum_{i=1}^{N_1} \frac{C_i}{a_0 - \alpha_i} e^{\Phi(\alpha_i, r) - \Phi(a_1, r)} \right].$$

The same solution can be obtained from a 1×1 Wronskian:

$$u = 2\partial_x^2 \log \left[e^{\Phi(a_1, r)} + \sum_{i=1}^{N_1} \frac{C_i}{a_0 - \alpha_i} e^{\Phi(\alpha_i, r)} \right] = 2\partial_x^2 \log \widetilde{F}.$$

Two examples with $N = 1$

Now suppose that

$$f_n(\alpha) = C_n\delta(\alpha - \alpha_n), \quad g_n(a) = \delta(a - a_n), \quad n = 1, \ldots, N,$$

where we assume that

$$a_1 > \cdots > a_N > \alpha_N > \cdots > \alpha_1, \quad C_1 > 0, \ldots C_N > 0.$$

In this case

$$u(r) = 2\partial_x^2 \log \sum_{I \subset \{1,\ldots,N\}} C_I \exp \Phi_I,$$

where

$$\Phi_I = \sum_{j=1}^{k} [\Phi(\alpha_{i_j}, r) - \Phi(a_{i_j}, r)],$$

and C_I is a multiple of a Cauchy determinant

$$C_I = C_{i_1} \cdots C_{i_k} \frac{\prod_{n=2}^{k} \prod_{m=1}^{n-1} (a_{i_n} - a_{i_m})(\alpha_{i_m} - \alpha_{i_n})}{\prod_{n=1}^{k} \prod_{m=1}^{k} (a_{i_n} - \alpha_{i_m})}.$$

The end

THANK YOU!