CR transversality of holomorphic maps into hyperquadrics

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Background

Let M be a connected smooth hypersurface in $\mathbb{C}^n$ near p. $n \geq 2$. The CR tangent space of M at p is given by:

$$T_p^{(1,0)}M = \{X \in T_pM : JX = iX\}.$$

Here J is the complex structure of M at p.

Regular coordinates of CR hypersurfaces

Let $(M, 0)$ be a germ of smooth CR hypersurface at 0. After a holomorphic change of coordinates, M is locally defined by

$$M = \left\{(z, w) \in \mathbf{C}^{n-1} \times \mathbf{C} : r = \Im w - \phi(z, \bar{z}, \Re w) = 0\right\},$$

where $\phi(0) = 0, d\phi(0) = 0$. See the book of Baouendi-Ebenfelt-Rothschild.

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Under the above regular coordinates (z, w),

$$T_0^{(1,0)} M = Span_{1 \leq j \leq n-1} \{ \frac{\partial}{\partial z_j} |_0 \}.$$

Examples of CR hypersurfaces: Levi-nondegenerate hypersurfaces

A smooth germ of a CR hypersurface M_ℓ in $\mathbb{C}^n$ is called a Levi-nondegenerate hypersurface of signature ℓ if it is locally defined by

$$M_\ell = \{(z, w) \in \mathbf{C}^{n-1} \times \mathbf{C} : r = \Im w - |z|_\ell^2 + O(3) = 0\}.$$

Here $|z|_\ell^2 = -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n-1} |z_j|^2$.

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Prototype - The hyperquadric in $\mathbb{C}^n$ of signature ℓ.

$$H_\ell^n = \left\{(z, w) \in \mathbf{C}^{n-1} \times \mathbf{C} : r = \Im w - |z|_\ell^2 = 0\right\}.$$

Question

Let M and $\tilde{M}$ be two connected smooth CR hypersurfaces in $\mathbf{C}^n$ and $\mathbf{C}^N$, respectively. $2 \leq n \leq N$. Let F be a smooth CR map with $F(M) \subset \tilde{M}$.

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Question:

Understand the geometric conditions on M and $\tilde{M}$ so that $F(M)$ intersects with $T^{(1,0)}\tilde{M}$ at generic position.

Definition of CR transversality

Definition

$F : (M, p) \to (\tilde{M}, F(p))$ is said to be CR transversal to $\tilde{M}$ at p if

$$dF(T_pM) \not\subset T^{(1,0)}_{F(p)}\tilde{M} \cup \overline{T^{(1,0)}_{F(p)}\tilde{M}}.$$

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When the CR map F extends holomorphically to a full neighborhood of p in $\mathbb{C}^n$, then F is CR transversal to $\tilde{M}$ at p iff

$$T^{(1,0)}_{F(p)}\tilde{M} + dF(T^{(1,0)}_p\mathbf{C}^n) = T^{(1,0)}_{F(p)}\mathbf{C}^N.$$

CR transversality in regular coordinates

Assume M and $\tilde{M}$ are defined by defining functions $r, \tilde{r}$ in regular coordinates (z, w) and $(\tilde{z}, \tilde{w})$, respectively. Let $F := (\tilde{f}, g)$ be a holomorphic map from a small neighborhood of $\mathbb{C}^n$ into $\mathbb{C}^N$ sending $(M, 0)$ into $(\tilde{M}, 0)$.

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$\iff \frac{\partial g}{\partial w}(0) \neq 0.$

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F is CR transversal to $\tilde{M}$ at 0
$\iff \frac{\partial g}{\partial w}(0) \neq 0.$

Notice that $\tilde{r} \circ F = a \cdot r$ for some smooth function a.
F is CR transversal to $\tilde{M}$ at 0
$\iff a(0) \neq 0.$

Equal dimensional case

$F : (M, p) \to (\tilde{M}, \tilde{p})$. $M, \tilde{M}$ are hypersurfaces in $\mathbb{C}^N$. F is not constant.

- Pinčuk, 1974, Siberian Math. J.
  $D, \tilde{D}$ strongly pseudoconvex in $\mathbb{C}^n$, $F : D \to \tilde{D}$ proper holomorphic, $F \in C^1(\bar{D}) \Rightarrow F$ is local biholomorphic. When F is a self holomorphic map between D, then it extends as a homeomorphism onto the boundary.
- Fornaess, 1978, Pacific J. Math.
  Let $D, \tilde{D}$ be C^2 bounded pseudoconvex, $F : D \to \tilde{D}$ biholomorphic and $F \in C^2(\bar{D}) \Rightarrow F : \bar{D} \to \bar{\tilde{D}}$ is diffeomorphic.
- Baouendi-Rothschild, 1990, J. Diff. Geom.
  $\tilde{M}$ is of finite type in the sense of Kohn-Bloom-Graham and F is of finite multiplicity $\Rightarrow F$ is CR transversal.
- Baouendi-Rothschild, 1993, Invent. Math.
  $M, \tilde{M}$ hypersurfaces of finite D'Angelo type at p and $\tilde{p}$, $\tilde{M}$ is minimally convex at $\tilde{p} \Rightarrow F$ is CR transversal.

Equal dimensional case, continued

$F : (M, p) \to (\tilde{M}, \tilde{p})$. $M, \tilde{M}$ are hypersurfaces in $\mathbb{C}^N$. F is not constant.

- Baouendi-Huang-Rothschild, 1995, Math. Res. lett.
  M essentially finite at all points, $Jac(F) \not\equiv 0$ and $F^{-1}(\tilde{p})$ is compact $\Rightarrow F$ is CR transversal.
- Huang-Pan, 1996, Duke. Math. J.
  $M, \tilde{M}$ real analytic minimal hypersurfaces $\Rightarrow$ the normal components of F is not flat.
- Lamel-Mir, 2006, Sci. China.
  M belongs to the class $\mathcal{C}$, $\tilde{M}$ is of finite D'Angelo map $\Rightarrow F$ is CR transversal.
- Ebenfelt-Son, 2012, Proceedings AMS.
  M is of finite type and F is of generic full rank $\Rightarrow F$ is CR transversal.

CR transversality between strictly pseudoconvex domains - Hopf Lemma

Let $F = (\tilde{f}, g)$ be a holomorphic map between two strictly pseudoconvex hypersurfaces $(M, 0) \subset \mathbb{C}^n$ and $(\tilde{M}, 0) \subset \mathbb{C}^N$. Assume

$$M = \left\{ (z, w) \in \mathbf{C}^{n-1} \times \mathbf{C} : r = \Im w - |z|^2 + O(3) = 0 \right\}$$

$$\tilde{M} = \left\{ (\tilde{z}, \tilde{w}) \in \mathbf{C}^{N-1} \times \mathbf{C} : \tilde{r} = \Im \tilde{w} - |\tilde{z}|^2 + O(3) = 0 \right\}$$

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$$\tilde{M} = \left\{(\tilde{z}, \tilde{w}) \in \mathbf{C}^{N-1} \times \mathbf{C} : \tilde{r} = \Im \tilde{w} - |\tilde{z}|^2 + O(3) = 0\right\}$$

Since $\Im g - |\tilde{f}|^2$ is a superharmonic function in $M^- : \left\{(z, w) \in \mathbf{C}^{n-1} \times \mathbf{C} : r = \Im w - |z|^2 + O(3) < 0\right\}$, by Hopf Lemma at 0,

$$\frac{\partial g}{\partial w}(0) = \frac{\partial \Im g}{\partial \Im w}(0) - i\frac{\partial \Re g}{\partial \Im w}(0) = \frac{\partial(\Im g - |\tilde{f}|^2)}{\partial \Im w}|_0 \neq 0.$$

(Notice $\frac{\partial \Re g}{\partial \Im w}(0) = -\frac{\partial \Im g}{\partial \Re w}(0) = 0$.)

Points of CR transversality are open and dense

Theorem (Baouendi-Ebenfelt-Rothschild, 2007, Comm. Ana. Geom.)

Let $M \subset \mathbb{C}^n$ be a germ of real-analytic hypersurface and U an open neighborhood of M in $\mathbb{C}^n$. Let $F : U \to \mathbb{C}^N$ is a holomorphic mapping with $F(M) \subset \tilde{M}$. Then either $F(U) \subset \tilde{M}$ or F is transversal to $\tilde{M}$ at $F(p)$ outside a proper real analytic subset if one of the following conditions holds:

- $\tilde{M} \subset \mathbb{C}^N$ *is a hyperquadric and* $N \leq 3(n - \nu_0(M)) - 2$.
- M *is holomorphically nondegenerate and* $\min(\nu^+(\tilde{M}), \nu^-(\tilde{M})) + \nu_0(\tilde{M}) \leq n - 2$.
- M *is holomorphically nondegenerate and* $N + \nu_0(\tilde{M}) \leq 2(n-1)$.

Examples of CR nontransversality

Example (Baouendi-Ebenfelt-Rothschild, 2007, Comm. Ana. Geom.)

$$M = H^2 = \left\{(z,w) \in \mathbf{C} \times \mathbf{C} : r = \Im w - |z|^2 = 0\right\};$$

$$\tilde{M} = H_1^5 = \left\{(z,w) \in \mathbf{C}^4 \times \mathbf{C} : \tilde{r} = \Im w + |z_1|^2 - \sum_{j=2}^{4} |z_j|^2 = 0\right\}.$$

We can verify that

$$F(z,w) = (iz + zw, -iz + zw, w, \sqrt{2}z^2, iw^2)$$

sends M into $\tilde{M}$ and $\tilde{r} \circ F = -2r^2$. F is nowhere CR tranversal on M.

CR Transversality of holomorphic maps between hyperquadrics of the same signature

A rigidity theorem of Baouendi-Huang:

Theorem (Baouendi-Huang, 2005, J. Diff. Geom.)

Let M be a small neighborhood of 0 in H^n_ℓ with $0 < \ell < \frac{n-1}{2}$, $n \geq 3$. Suppose F is a holomorphic map from a neighborhood U of M in $\mathbb{C}^n$ into $\mathbb{C}^N$ with $F(M) \subset H^N_\ell, N \geq n$ and $F(0) = 0$. Then either $F(U) \subset H^N_\ell$ or F is linear fractional.

CR Transversality of holomorphic maps between hyperquadrics of the same signature

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Theorem (Baouendi-Huang, 2005, J. Diff. Geom.)

Let M be a small neighborhood of 0 in H_ℓ^n with $0 < \ell < \frac{n-1}{2}$, $n \geq 3$. Suppose F is a holomorphic map from a neighborhood U of M in $\mathbb{C}^n$ into $\mathbb{C}^N$ with $F(M) \subset H_\ell^N, N \geq n$ and $F(0) = 0$. Then either $F(U) \subset H_\ell^N$ or F is linear fractional.

Under the assumption in Baouendi-huang, either $F(U) \subset H_\ell^N$ or F is CR transversal to H_ℓ^N.

Conjecture of Baouendi-Huang

Conjecture (Baounedi-Huang, 2005):

Let $M_1 \subset \mathbf{C}^n$ and $M_2 \subset \mathbf{C}^N$ be two (connected) Levi non-degenerate real analytic hypersurfaces with the same signature $\ell > 0$. Here $3 \leq n < N$. Let F be a holomorphic map defined in a neighborhood U of M_1, sending M_1 into M_2. Then either F is a local CR embedding from M_1 into M_2 or F is totally degenerate in the sense that it maps a neighborhood U of M_1 in $\mathbf{C}^n$ into M_2.

Main theorem

Theorem (Huang-Z., to appear in Abel Symposia)

Let M_ℓ be a real analytic Levi non-degenerate hypersurface of signature ℓ in $\mathbf{C}^n$ with $n \geq 3$ and $0 \in M_\ell$. Suppose that F is a holomorphic map in a small neighborhood U of $0 \in \mathbf{C}^n$ such that

$$F(M_\ell \cap U) \subset H_\ell^N$$

with $N - n < \frac{n-1}{2}$. If $F(U) \not\subset H_\ell^N$, then F is CR transversal to M_ℓ at 0, or equivalently, F is a CR embedding from a small neighborhood of $0 \in M_\ell$ into H_ℓ^N.

Normalization of a CR transversal map

Assume F is not CR transversal to M_ℓ at 0 and $F(U) \not\subset H_\ell^N$. By Baouendi-Ebenfelt-Rothschild, we can choose a sequence $\{p_j\} \in M_\ell$ such that $p_j \to 0$ and F is CR transversal at each p_j with $j \geq 1$. Write $q_j := F(p_j)$. Now for each j, assume M_ℓ and $\tilde{M}_\ell$ are both in regular coordinates at p_j and $F(p_j)$ WLOG. We do normalizations on F at p following Huang (1999) and Baouendi-Huang (2005).
1)Consider $F_{p_j} := \tau_{F(p_j)} \circ F \circ \sigma_{p_j} = (\tilde{f}_{p_j}, \tilde{g}_{p_j})$, where $\sigma_p \in Aut(H_\ell^n)$ and $\tau_{F(p)} \in Aut(H_\ell^N)$ such that $\sigma_p(0) = p$ and $\tau_{F(p)}(F(p)) = 0$. We have

$$\tilde{f}_{p_j} = \lambda_j z U_j + \vec{a}_j w + O(|(z,w)|^2)$$
$$\tilde{g}_{p_j} = \lambda_j^2 w + O(|(z,w)|^2)$$

with $\lambda_j \to 0$.

Normalization of a CR transversal map, continued

2) Consider $F^{\sharp}_{p_j} = T_j \circ F_{p_j} = (f^{\sharp}_{p_j}, \phi^{\sharp}_{p_j}, g^{\sharp}_{p_j})$ where $T_j \in Aut_0(H^N_\ell)$ and

$$T_j(\tilde{z}, \tilde{w}) = \frac{(\lambda_j^{-1}(\tilde{z} - \lambda_j^{-2}\vec{a}_j\tilde{w})\tilde{U}_j^{-1}, \lambda_j^{-2}\tilde{w})}{1 + 2i\langle \tilde{z}, \lambda_j^{-2}\overline{\vec{a}_j}\rangle_\ell + \lambda_j^{-4}(r_j - i|\vec{a}_j|_\ell^2)\tilde{w}}.$$

$r_j = \frac{1}{2}\Re\{(g_j)''_{ww}(0)\}$. We have

$$f^{\sharp}_{p_j}(z, w) = z + \sum_{k=3}^{\infty} {f^{\sharp}_{p_j}}^{(k)}(z, w),$$
$$\phi^{\sharp}_{p_j}(z, w) = \sum_{k=2}^{\infty} {\phi^{\sharp}_{p_j}}^{(k)}(z, w),$$
$$g^{\sharp}_{p_j}(z, w) = w + \sum_{k=5}^{\infty} {g^{\sharp}_{p_j}}^{(k)}(z, w).$$

Here for a holomorphic function f, $f^{(k)}$ represent the weighted degree k term in the power series expansion of f.

Normalization of a CR transversal map, continued

2) Consider $F^{\sharp}_{p_j} = T_j \circ F_{p_j} = (f^{\sharp}_{p_j}, \phi^{\sharp}_{p_j}, g^{\sharp}_{p_j})$ where $T_j \in Aut_0(H^N_\ell)$ and

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$r_j = \frac{1}{2}\Re\{(g_j)''_{ww}(0)\}$. We have

$$f^{\sharp}_{p_j}(z, w) = z + \sum_{k=3}^{\infty} {f^{\sharp}_{p_j}}^{(k)}(z, w),$$

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Here for a holomorphic function f, $f^{(k)}$ represent the weighted degree k term in the power series expansion of f.

Question: For each k, what happens for ${f^{\sharp}_{p_j}}^{(k)}, {\phi^{\sharp}_{p_j}}^{(k)}, {g^{\sharp}_{p_j}}^{(k)}$ when $p_j \to 0$?

Huang's Lemma

Lemma (Huang, 1999, J. Diff. Geom.)

Let $\{\phi_j\}_{j=1}^{n-1}$ *and* $\{\psi_j\}_{j=1}^{n-1}$ *be two families of holomorphic functions in* $\mathbf{C}^n$*. Let* $B(z,\xi)$ *be a real-analytic function in* (z,ξ)*. Suppose that*

$$\sum_{j=1}^{n-1} \phi_j(z)\psi_j(\xi) = B(z,\xi)\langle z,\xi\rangle_\ell.$$

Then $B(z,\xi) = \sum_{j=1}^{n-1} \phi_j(z)\psi_j(\xi) = 0$*.*

A quantitative version of Huang's Lemma

Lemma

Let $\{\phi_j\}_{j=1}^{n-1}$ and $\{\psi_j\}_{j=1}^{n-1}$ be two families of holomorphic polynomials of degree k and m in $\mathbf{C}^n$, respectively. Let $H(z,\xi), B(z,\xi)$ be two polynomials in (z,ξ). Suppose that

$$\sum_{j=1}^{n-1} \phi_j(z)\psi_j(\xi) = H(z,\xi) + B(z,\xi)\langle z,\xi\rangle_\ell$$

and $\|H\| \leq C$. Then $\|B\| \leq \tilde{C}$ and $\|\sum_{j=1}^{n-1} \phi_j(z)\psi_j(\xi)\| \leq \tilde{C}$ with $\tilde{C}$ dependent only on (C,k,m,n).

Definition:

$$\|\sum_{|\alpha|\leq k} a_\alpha z^\alpha\| := max_{|\alpha|\leq k}\{|a_\alpha|\}.$$

A quantitative version of Huang's Lemma, continued

Lemma

Let $\{\phi_{jr}\}_{j=1}^{n-1}$ *and* $\{\psi_{jr}\}_{j=1}^{n-1}$ *be two families of holomorphic polynomials in* $\mathbf{C}^n$, $1 \leq r \leq m$. *Let* $H(z,\xi), B(z,\xi)$ *be two polynomials in* (z,ξ). *Suppose that*

$$\sum_{r=1}^{m}\left(\sum_{j=1}^{n-1}\phi_{jr}(z)\psi_{jr}(\xi)\right)\langle z,\xi\rangle_\ell^r = H(z,\xi) + B(z,\xi)\langle z,\xi\rangle_\ell^{m+1}$$

and $\|H\| \leq C$. *Then* $\|B\| \leq \tilde{C}$ *and* $\|\sum_{j=1}^{n-1}\phi_{jr}(z)\psi_{jr}(\xi)\| \leq \tilde{C}$ *for all* $1 \leq r \leq m$ *with* $\tilde{C}$ *dependent only on* (C, n, m) *and the degrees of* ϕ_{jr}, ψ_{jr} *for all* $1 \leq r \leq m$.

Perturbation of the map

Making use of the quantitative version of Huang's Lemma, we show when $N < 2n-1$, for each k,

$$\|{f^{\sharp}_{p_j}}^{(k)}\|, \|{\phi^{\sharp}_{p_j}}^{(k)}\|, \|{g^{\sharp}_{p_j}}^{(k)}\| \leq C_k.$$

Hence by a result of Meylan-Mir-Zaitsev, we obtain

Theorem (Huang-Z., 2014)

Let M_ℓ be a germ of a smooth Levi non-degenerate hypersurface at 0 of signature ℓ in $\mathbf{C}^n$, $n \geq 3$. Suppose that there exists a holomorphic map F in a neighborhood U of 0 in $\mathbf{C}^n$ sending M_ℓ into H^N_ℓ but $F(U) \not\subset H^N_\ell$, $N < 2n-1$. Then M_ℓ is CR embeddable into H^N_ℓ near 0. Equivalently, there exists a holomorphic map $\tilde{F} : M_\ell \to H^N_\ell$ near 0, which is CR transversal to M_ℓ at 0.

Proof of the main theorem

Assume by contradiction that F neither is CR transversal to M_ℓ at 0 nor sends U into H_ℓ^N. Then there exists a CR immersion F^* sending M_ℓ into H_ℓ^N by the perturbation theorem.
On the other hand, by a rigidity result of Ebenfelt-Huang-Zaitsev, when the codimension is less than $\frac{n-1}{2}$, there exists an automorphism T of H_ℓ^N such that near $p_j \approx 0$, and hence at all points in M_ℓ near the origin,

$$F = T \circ F^*.$$

Since T extends to an automorphism of the projective space $\mathbf{P}^N$ and $T(0) = 0$, F must be CR transversal at 0. This is a contradiction. □

Thank you!