Flat solutions to the Cauchy-Riemann Equations

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Motivation - two Unique continuation Property (UCP) problems

Definition: A smooth function (or map) f is said to be *flat* (at 0) if $D^\alpha f(0) = 0$ for all multi-indices α.

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$D_R := \{z \in \mathbf{C} : |z| = R\}$. $B_R := \{z \in \mathbf{C}^n : |z| = R\}$.

Theorem (Chanillo-Sawyer)

Let $V \in L^2(D_R)$ and $u : D_R \subset \mathbb{R}^2 \to \mathbb{R}^N$ be smooth. If $|\Delta u| \leq V|\nabla u|$, then UCP holds, i.e., $u \equiv 0$ on D_R whenever u is flat.

Theorem (Pan)

Let $V \in L^2(D_R)$ and $v : D_R \subset \mathbb{C} \to \mathbb{C}^M$ be smooth. If $|\bar\partial v| \leq V|v|$, then UCP holds, i.e., $v \equiv 0$ on D_R whenever v is flat.

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What happens in the sense of germs (where f cannot be trivially 0 near 0)?

Criterion in $\mathbb{C}$

Lemma

Let f be flat at $0 \in \mathbb{C}$. The following two statements are equivalent:
1) $\bar{\partial} u = f d\bar{z}$ has a flat solution locally.
2) There exists some neighborhood U of 0 such that the following series

$$\sum_{n=0}^{\infty} \left(\int_U \frac{f(\xi)}{\xi^{n+1}} d\bar{\xi} \wedge d\xi \right) z^n$$

is holomorphic near 0.

Sketch of the proof

Denote the *Cauchy-Green operator* by $Tf(z) := \frac{-1}{2\pi i}\int_{D_R} \frac{f(\zeta)}{\zeta - z} d\bar{\zeta} \wedge d\zeta$. Then $\bar{\partial} Tf = f d\bar{z}$ on D_R.

Higher order derivative formulas of T on D_R:

Theorem (Pan, preprint)

Let $f \in C^{k+\alpha}(D_R)$ with $0 < \alpha < 1$ and $k \in \mathbb{Z}^+ \cup \{0\}$. Then

$$\partial^{k+1} T(f)(z) = \frac{-k!}{2\pi i}\int_{D_R} \frac{f(\zeta) - P_k(\zeta, z)}{(\xi - z)^{k+2}} d\bar{\zeta} \wedge d\zeta$$

on D_R, where $P_k(\zeta, z)$ is the Taylor expansion of f at z of degree k.

See [Liu-Pan-Z., 2015, preprint] for the higher order derivative formulas of T on general domains.

Positive examples in $\mathbb{C}$

Example

Let $\varphi \in C^\infty(\mathbb{R}, \mathbb{C})$ be flat at 0 and g be harmonic on D. Then $\bar{\partial} u(z) = \varphi(|z|)g(z)d\bar{z}$ always has a flat solution locally.

Counter-examples in $\mathbb{C}$

The construction is essentially motivated by Rosay and Coffman-Pan.
s: a nondecreasing function on $\overline{\mathbb{R}^+}$, $s = 0$ in $[0, \frac{1}{4}]$, $0 < s < 1$ on $(\frac{1}{4}, \frac{3}{4})$ and $s = 1$ on $[\frac{3}{4}, \infty)$;
$\{r_n\}_{n=1}^{\infty}$: a decreasing positive sequence, $\lim_{n\to\infty} r_n = 0$. $\Delta r_n := r_n - r_{n+1}$, annuli $A_n := \{z \in \mathbb{C} : r_{n+1} \le |z| \le r_n\}$;
$\{p(n)\}_{n=0}^{\infty}$: an increasing positive integer sequence with $p(0) = 0$;
$\{F(n)\}_{n=0}^{\infty}$: a positive sequence with $F(0) = 1$.
Let $g_n(z) = F(n)z^{p(n)}$, $\mathcal{X}_n = s(\frac{|\cdot| - r_{n+1}}{\Delta r_n}) : A_n \to \mathbb{R}$, and

$$f(z) = \begin{cases} g_n(z), & z \in A_n \text{ for odd n}, \\ \mathcal{X}_n(z)g_{n-1}(z) + (1 - \mathcal{X}_n(z))g_{n+1}(z), & z \in A_n \text{ for even n}, \\ 0, & z = 0. \end{cases}$$

Lemma of Coffman-Pan

Lemma (Coffman-Pan)

If $\frac{(\Delta r_n/r_n)}{(\Delta r_{n+2})/(r_{n+2})}$ *is a bounded sequence and for each integer* $k \geq 0$*,*

$$\lim_{n\to\infty} \frac{F(n+1)(p(n+1))^k r_n^{p(n+1)-4k}}{(\Delta r_n/r_n)^k} = 0,$$

then f *is smooth and flat at the origin.*

The Family **S**

Denote by **S** the set of functions f such that $\frac{(\Delta r_n/r_n)}{(\Delta r_{n+2})/(r_{n+2})}$ is bounded,

$$\lim_{n\to\infty} \frac{F(n+1)(p(n+1))^k r_n^{p(n+1)-4k}}{(\Delta r_n/r_n)^k} = 0,$$

as well as either one of the following conditions:

$$\lim_{n\to\infty} \sqrt[p(n)]{F(n)\Delta r_n r_{n+1}} = \infty,$$
$$\lim_{n\to\infty} \sqrt[p(n)]{F(n)(\Delta r_{n-1})^2} = \infty,$$
$$\lim_{n\to\infty} \sqrt[p(n)]{F(n)\Delta r_{n-1} r_n} = \infty,$$
$$\lim_{n\to\infty} \sqrt[p(n)]{F(n)(\Delta r_{n+1})^2} = \infty,$$
$$\lim_{n\to\infty} \sqrt[p(n)]{F(n)\Delta r_{n+1} r_{n+2}} = \infty.$$

Examples in **S**

Example (Rosay)

$R = 1,\ p(n) = n,\ r_n = 2^{-n+1},\ F(n) = 2^{n^2/2}.$

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Example

$R = 1$. $p(n), t(n)$ and $q(n)$ are polynomials of degree d_p, d_t and d_q with positive leading coefficients, $t(1) = 0$, $d_q > d_p$, $d_q > d_t$ and $d_q < d_p + d_t$. Let $r_n := 2^{-t(n)}$, $F(n) := 2^{q(n)}$.

The main theorems

Theorem

For every $f \in \mathbf{S}$, there does not exist a flat smooth u such that $\bar{\partial} u = f d\bar{z}$ near the origin.

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Theorem

There exists a family of germs of $\bar{\partial}$-closed (0,1) forms, flat at $0 \in \mathbb{C}^n$, such that for every f in this family, the Cauchy-Riemann equation $\bar{\partial} u = f$ has no flat solution in the sense of germs.

Hörmander's L^2 theory

Theorem (Hörmander, Acta. Math., 1965)

Let Ω be a bounded pseudoconvex open set in $\mathbb{C}^n$, Let δ be the diameter of Ω, and let ϕ be a plurisubharmonic function in Ω. For every $\bar{\partial}$-closed $f \in L^2_{(0,q)}(\Omega, \phi), q > 0$, one can find $u \in L^2_{(0,q-1)}(\Omega, \phi)$ satisfying $\bar{\partial} u = f$ in Ω and

$$q \int_\Omega |u|^2 e^{-\phi} dV \leq e\delta^2 \int_\Omega |f|^2 e^{-\phi} dV.$$

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When $q = 1$, a *minimal solution* to $\bar\partial u = f$ on Ω is the solution that is orthogonal to the space of holomorphic functions with respect to $L^2(\Omega, \phi)$ norm.

Is the restriction of a minimal solution minimal?

Ω_1, Ω_2: smooth bounded pseudoconvex domains, $\Omega_2 \subset \Omega_1$;
ϕ: a bounded plurisubharmonic function in Ω_1;
f: a $\bar{\partial}$-closed (0,1) form in Ω_1.
Consider the minimal solution u_1 to

$$\bar{\partial} u = f, \quad \Omega_1$$

with respect to $L^2(\Omega_1, \phi)$ norm and the minimal solution u_2 to

$$\bar{\partial} u = f|_{\Omega_2}, \quad \Omega_2$$

with respect to $L^2(\Omega_2, \phi|_{\Omega_2})$ norm.

Question: Is $u_2 = u_1|_{\Omega_2}$?

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In general, No! Examples?

Examples: Let $\tilde{f} \in \mathbf{S}$ and consider $f(z) := \tilde{f}(z_1) d\bar{z}_1$.

Conclusion: For every f above, any given bounded plurisubharmonic weight function ϕ on B_1 and positive decreasing sequence $r_n (< 1) \to 0$, the minimal solution u_n to $\bar{\partial} u = f|_{B_{r_n}}$ on B_{r_n} with respect to $L^2(B_{r_n}, \phi|_{B_{r_n}})$ norm is not the restriction of u_1 onto B_{r_n}.

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Conclusion: For every f above, any given bounded plurisubharmonic weight function ϕ on B_1 and positive decreasing sequence $r_n (< 1) \to 0$, the minimal solution u_n to $\bar{\partial} u = f|_{B_{r_n}}$ on B_{r_n} with respect to $L^2(B_{r_n}, \phi|_{B_{r_n}})$ norm is not the restriction of u_1 onto B_{r_n}.

Sketch of the proof: If not, then for each N, when n is large enough,

$$\int_{B_{r_n}} |u_1|^2 dV \leq C \int_{B_{r_n}} |u_n|^2 e^{-\phi} dV \leq C r_n^2 \int_{B_{r_n}} |f(z_1)|^2 e^{-\phi} dV$$
$$\leq C r_n^2 \int_{B_{r_n}} |f(z_1)|^2 dV \leq C r_n^N.$$

$\Rightarrow$ u_1 is flat. Contradiction!

Positive examples

Inspired by an example of Z. Błocki, we have

Example: Let f_j and g be holomorphic in B_R such that $g(0) = 0$ and $\frac{\partial g}{\partial z_j} = f_j$ in B_R. Then given any bounded and radially symmetric plurisubharmonic weight ϕ on B_R, $u(z) = \overline{g(z)}|_{B_r}$ is the minimal solution to $\bar{\partial} u(z) = \overline{f_j(z)} d\bar{z}_j|_{B_r}$ in B_r in $L^2(B_r, \phi|_{B_r})$ norm for every $r \leq R$.

Thank you!